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EXAMINING THE NONLINEAR INTERACTION OF SURFACE SH WAVES AND THEIR EFFECTS ON A WAVEGUIDE SURFACE

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Waves in elastic media find applications in various fields. One prominent area is **seismology**, where the study of elastic waves helps us understand the Earth's interior. These waves travel through the Earth and provide valuable information about its composition and structure.

In **engineering**, the understanding of elastic waves is crucial in non-destructive testing (NDT) techniques. Ultrasonic testing, for example, relies on the propagation of elastic waves to detect flaws or abnormalities in materials without causing damage.

Acoustic waves in elastic media also play a role in **medical imaging**, particularly in ultrasound technology. Doctors use ultrasound to visualize internal organs and monitor the development of fetuses during pregnancy.

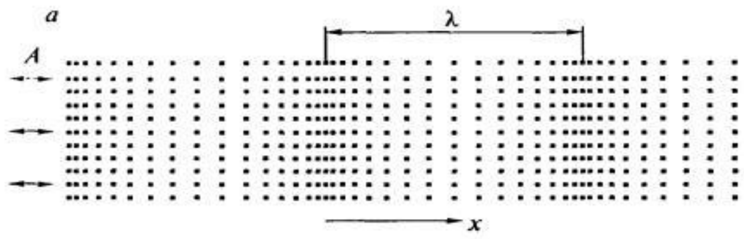
Additionally, elastic waves are studied in **materials science** for characterizing the mechanical properties of materials. This knowledge is essential in fields like civil engineering, where it helps in designing structures that can withstand various stresses.

So, whether it's exploring the Earth's subsurface, ensuring the structural integrity of materials, or peering inside the human body, the study of waves in elastic media and interaction of several simultaneously propagating waves has a wide range of practical applications.

Wave equation: $$\Delta u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

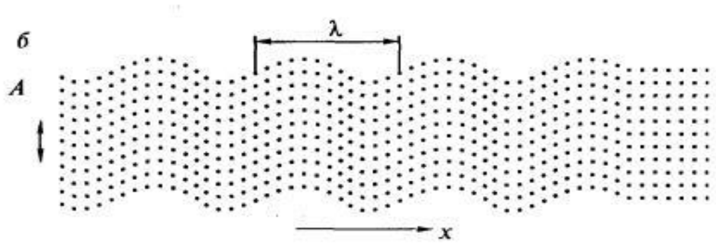
Primary wave (P-wave)

Compressional wave that causes particles in the material to move in the same direction as the wave is propagating. It is the fastest seismic wave and can travel through solids, liquids, and gases.



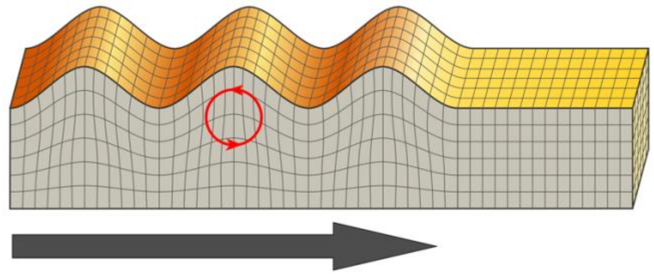
Shear wave (S-wave)

A transverse wave that causes particles to move perpendicular to the direction of the wave. S-waves cannot travel through liquids or gases, only through solids.



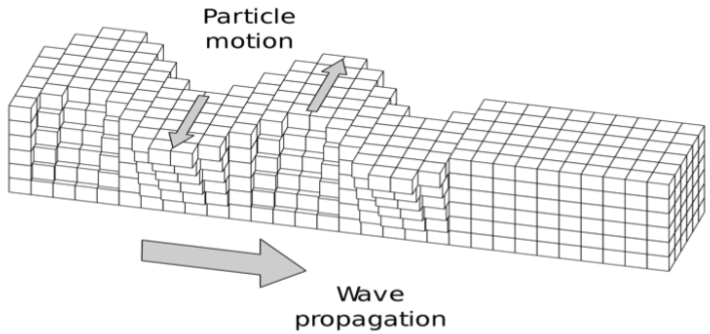
Raileigh waves (P-SV type of motion)

Lord Rayleigh. On Waves Propagated along the Plane Surface of an Elastic Solid (англ.) // Proc. London Math. Soc. : journal. — 1885. — Vol. s1—17, no. 1. — P. 4—11



Love waves (SH type of motion)

Love A. E. H. Some problems of geodynamics. First published in 1911 by the Cambridge University Press and published again in 1967 by Dover, New York, USA. (Chapter 11: Theory of the propagation of seismic waves).



The analysed model is based on the tensor representation of the cubic elastic potential $\widetilde{U}$

$$\widetilde{U} = \frac{1}{2}\tilde{c}_{jqrk}\tilde{\varepsilon}_{jq}\tilde{\varepsilon}_{rk} + \frac{1}{6}\tilde{c}_{jqrklm}\tilde{\varepsilon}_{jq}\tilde{\varepsilon}_{rk}\tilde{\varepsilon}_{lm} \quad (j, q, r, k, l, m = \overline{1,3}) \tag{1.1}$$

with finite elastic deformations

$$\tilde{\varepsilon}_{rk} = \frac{1}{2}\left(\tilde{u}_{r,k} + \tilde{u}_{k,r} + \tilde{u}_{l,r}\tilde{u}_{l,k}\right), \tag{1.2}$$

where $\tilde{u}_{r,k} = \partial\tilde{u}_r / \partial\tilde{x}_k$, $\tilde{u}_r$ — components of the vector of elastic wave displacements.
Mechanical stresses σ_{jd} (1.3) are used in the form of decomposition into linear and nonlinear components (1.4-1.5):

$$\sigma_{jd} = {}^{\partial U}\!/_{\partial u_{j,d}} \qquad (j, d = \overline{1,3}). \tag{1.3}$$

$$\sigma_{jd} = \sigma_{jd}^{(0)} + \sigma_{jd}^{(1)}, \tag{1.4}$$

$$\sigma_{jd}^{(0)} = \left[c_{jqrk}u_{r,k}\right]\delta = \hat{\sigma}_{jd}^{(0)}\delta,$$

$$\sigma_{jd}^{(1)} = \left[\frac{1}{2}c_{jqrk}u_{l,r}u_{l,k} + c_{pdrk}u_{j,p}u_{r,k+}\frac{1}{2}c_{jdrklm}u_{r,k}u_{l,m}\right]\delta^2 = \hat{\sigma}_{jd}^{(1)}\delta^2 \tag{1.5}$$

The equation of wave motion is used:

$$\left({}^{\tilde{\rho}R_*^2}\!/_{c_*}\right)\delta\ddot{u}_j - \sigma_{jd,d}^{(0)} = \sigma_{jd,d}^{(1)} \qquad (j, d = \overline{1,3}). \tag{1.6}$$

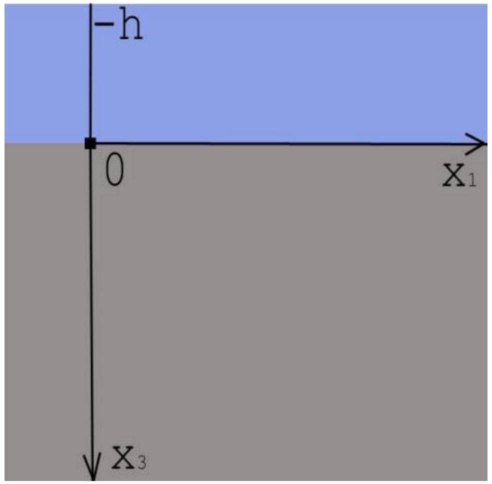
The **waveguide material** is **anisotropic cubic crystal of *m3m* class** which characterized by:

- Density ρ;
- Elastic constants of second order $c_{11}, c_{12}, c_{44;}$
- Elastic constants of third order $c_{111}, c_{112}, c_{144}, c_{155}, c_{123}, c_{456}$.

Normalization of physical and mechanical characteristics

The components $\tilde{u}_j^{(p)}$ of the vector of elastic wave displacements in the switching to the dimensionless normalized characteristics $u_j^{(p)}$ are normalized by the maximum wave amplitude $u_* = \max\limits_{\{x_1,x_2,x_3,t,j\}} |\tilde{u}_j(x_1,x_2,x_3,t)|$. Mechanical stresses and elastic constants are referred to the normalizing parameter $c_* = 10^{10} Pa$. Thus $x_j = \tilde{x}_j h_*^{-1}$, $u_j = \tilde{u}_j u_*^{-1}$, $c_{ij}^{(p)} = \tilde{c}_{ij}^{(p)}/c_*$, $c_{ijl}^{(p)} = \tilde{c}_{ijl}^{(p)}/c_*$. Value $\delta \ll 1$ for low intensity elastic waves is a small parameter.

Problem statement



The waveguide consists of a thin elastic layer V_1 lying on a half-space V_2. Where
$V_1 = \{-\infty < x_1,\ x_2 < \infty,\ -1 \le x_3 \le 0\}$,
$V_2 = \{-\infty < x_1,\ x_2 < \infty,\ 0 < x_3 < \infty\}$.

Boundary conditions:

$$\left(u_j^{(1)}\right)_{x_3=-1} = 0,$$

$$\left(\sigma_{3j}^{(1)}\right)_{x_3=0} = \left(\sigma_{3j}^{(2)}\right)_{x_3=0},\ \left(u_j^{(1)}\right)_{x_3=0} = \left(u_j^{(2)}\right)_{x_3=0}\ \ (j=\overline{1,3});$$

And condition of wave motions localization $\left(u_2^{(2)}\right)_{x_3\to\infty} \to 0.$

The approach used in this work is based on the concept of determining the characteristics of the investigated wave field as the first two terms in the representations of the vector functions of the wave strength in the component V_p by a segment of a series in powers of a small parameter δ:

$$u_j^{(p)} = u_j^{(p,l)} + \delta u_j^{(p,n)} \ (j = \overline{1,3}), \tag{1.7}$$

where in the considered case of anharmonic perturbations analysis for linear generalized surface SH waves $u_1^{(p,l)} \equiv u_3^{(p,l)} \equiv 0$. When this representation is substituted into the equations of motion and boundary conditions, as well as with the subsequent selection of terms of the same order of smallness in the parameter δ, it's arisen two-stage problems of determining the components $u_j^{(p,l)}$ and $u_j^{(p,n)}$.

At the first stage, a solution is constructed for each single problem on the propagation of a generalized linear wave. We consider a homogeneous spectral problem for the normalized displacement function $u_2^{(p,l)}$ of the wave:

$$\sigma_{2j,j}^{(p,l)} - \rho_p \ddot{u}_2^{(p,l)} = 0, \tag{1.8}$$

$$\ddot{u}_2^{(1,l)}{}_{x_3=-1} = 0, \ \ (\sigma_{32}^{(2,l)})_{x_3=0} = (\sigma_{32}^{(1,l)})_{x_3=0}, \ \ (u_2^{(2,l)})_{x_3=0} = (u_2^{(1,l)})_{x_3=0} \ \ (p = 1,2).$$

As a result of this spectral problem analysis, representations for linear approximations of the studied waves with a dimensionless amplitude parameter $u_2^{(0)}$ and a frequency Ω_m are presented:

$$u_2^{(1,l)} = u_2^{(0)}(\cos\big(\alpha^{(1)}x_3\big) - c_{44}^{(2)}\,\alpha^{(2)}\,\sin(\alpha^{(1)}x_3))/(c_{44}^{(1)}\alpha^{(1)})e^{-i(\omega t - kx_1)},$$

$$u_2^{(2,l)} = u_2^{(0)} e^{-\alpha^{(2)}x_3} e^{-i(\omega t - kx_1)}, \tag{1.9}$$

where $\alpha^{(1)} = ((\Omega_1/c_{44}^{(1)})^{1/2} - k^2)^{1/2}$, $\alpha^{(2)} = (k^2 - (\Omega_2/c_{44}^{(2)})^{1/2})^{1/2}$, $\Omega_j = (\rho_j R_*^2 \omega^2/c_*)^{1/2}$.

On the second stage, solutions of inhomogeneous problems for the amplitude characteristics of the second harmonics with frequencies $2\omega_m$ $(m = \overline{1,2})$ and harmonics of the "combination" type with frequency $(\omega_1 + \omega_2)$ are constructed. Components of the complex vector of wave displacements $u_{j+}^{(p,n)}$ $(j = 1; 3)$ for the second harmonic of "combination" type with frequency $\omega_1 + \omega_2$, provided that the anharmonic perturbation in the form of P-SV wave with frequency $\omega_1 + \omega_2$ and wave number $k_1 + k_2$ does not belong to the branch of the spectrum of generalized Rayleigh type surface waves in the considered waveguide, have the structure:

$$u_{1+}^{(1,n)} = (\lambda_{11+} \cos\left(\varsigma_{1+}^{(1)} x_3\right) + \lambda_{12+} \cos\left(\varsigma_{2+}^{(1)} x_3\right) + \mu_{11+} \sin\left(\varsigma_{1+}^{(1)} x_3\right) + \mu_{12+} \sin\left(\varsigma_{2+}^{(1)} x_3\right) +$$
$$+A_{1131} \cos\left(\left(\alpha_1^{(1)} + \alpha_2^{(1)}\right) x_3\right) + A_{1132} \sin\left(\left(\alpha_1^{(1)} + \alpha_2^{(1)}\right) x_3\right) + A_{1133} \cos\left(\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) x_3\right) +$$
$$+A_{1134} \sin(\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) x_3))\, E_{12}, \tag{1.10}$$

$$u_{3+}^{(1,n)} = (\lambda_{31+} \sin\left(\varsigma_{1+}^{(1)} x_3\right) + \lambda_{32+} \sin\left(\varsigma_{2+}^{(1)} x_3\right) + \mu_{31+} \cos\left(\varsigma_{1+}^{(1)} x_3\right) + \mu_{32+} \cos\left(\varsigma_{2+}^{(1)} x_3\right) +$$
$$+A_{3131} \sin\left(\left(\alpha_1^{(1)} + \alpha_2^{(1)}\right) x_3\right) + A_{3132} \cos\left(\left(\alpha_1^{(1)} + \alpha_2^{(1)}\right) x_3\right) + A_{3133} \sin\left(\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) x_3\right) +$$
$$+A_{3134} \cos(\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) x_3))\, E_{12},$$

$$u_{1+}^{(2,n)} = (\beta_{11+} \exp\,(\varsigma_{1+}^{(2)} x_3) + \beta_{12+} \exp\,(\varsigma_{2+}^{(2)} x_3) + A_{1231} \exp\,(2i(\alpha_1^{(2)} + \alpha_2^{(2)}) x_3)) E_{12},$$
$$u_{3+}^{(2,n)} = (\beta_{31+} \exp\,(\varsigma_{1+}^{(2)} x_3) + \beta_{32+} \exp\,(\varsigma_{2+}^{(2)} x_3) + A_{3231} \exp\,(2i(\alpha_1^{(2)} + \alpha_2^{(2)}) x_3)) E_{12},$$

where $E_{12}(t, x_1, \omega_m, k_m) = \exp(-i(\omega_1 + \omega_2)t - (k_1 + k_2)x_1)$.

Numerical investigations

Numerical studies are implemented for a waveguide in the form of a sodium chloride (NaCl) layer rigidly fixed from the upper surface and ideally in contact along the lower surface with a halfspace of silicon (Si) (Fig 1 - 2). Another waveguide combination is described on Fig. 3-4 for Germanium layer on a silicon.

For these shading images, an increase in values corresponds to a transition from darker tones to lighter ones.

Figures 1 and 3 respectively, describe the distributions of amplitude functions $|u_{1+}^{(n)}|/u_0^2$ and $|u_{3+}^{(n)}|/u_0^2$, depending on the combinations of normalized dimensionless frequency parameters $\Omega_{*m} = (\rho_* \omega_m^2 h^2 / c_*)^{1/2}$ for two SH waves with circular frequencies ω_1 and ω_2.

Figures 2 and 4, in its turn, describe the distributions of amplitude characteristics $|u_{1+}^{(n)}|/u_0^2$ and $|u_{3+}^{(n)}|/u_0^2$ for two SH waves, the first of which has a frequency Ω_{*1} and belongs to the first mode of the dispersion spectrum, and the second frequency Ω_{*2} belongs to the second mode of the dispersion spectrum.

The presented frequency distributions give an idea of the intensity measure and frequency regions of the multilevel nonlinear anharmonic interaction of generalized surface SH waves in case of outer rigid fixation of waveguide.

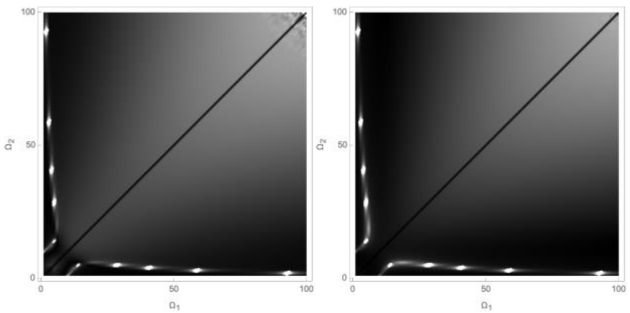


Fig. 1. The distributions of amplitude functions $|u_{1+}^{(n)}|/u_0^2$ and $|u_{3+}^{(n)}|/u_0^2$ depending on value $\Omega_j = \overline{0{,}10}$ at $x_3 = 0$ for the NaCl-Si waveguide.

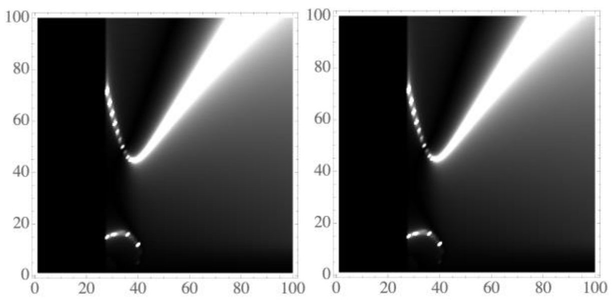


Fig. 2. The distributions of amplitude functions $|u_{1+}^{(n)}|/u_0^2$ and $|u_{3+}^{(n)}|/u_0^2$ depending on value $\Omega_1 = \overline{0,10}$ and $\Omega_2 = \overline{2.8,10}$ at $x_3 = 0$ for the NaCl-Si waveguide.

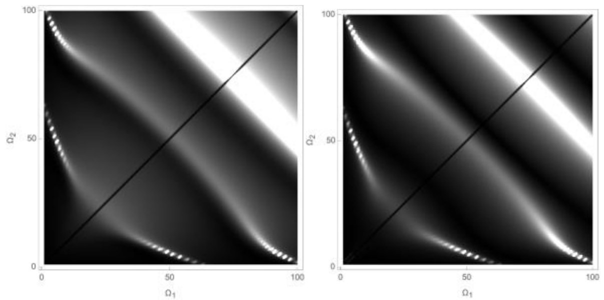


Fig. 3. The distributions of amplitude functions $\left|u_{1+}^{(n)}\right|/u_0^2$ and $\left|u_{3+}^{(n)}\right|/u_0^2$ depending on value $\Omega_j = \overline{0,10}$ at $x_3 = 0$ for the Ge-Si waveguide.

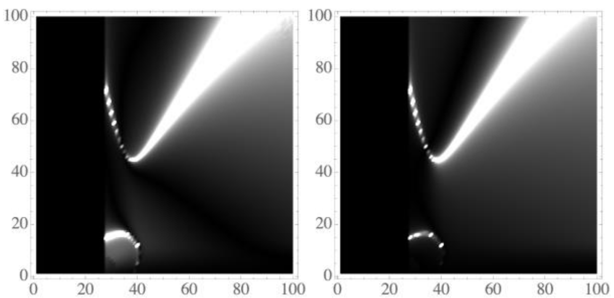


Fig. 4. The distributions of amplitude functions $|u_{1+}^{(n)}|/u_0^2$ and $|u_{3+}^{(n)}|/u_0^2$ depending on value $\Omega_1 = \overline{0,10}$ and $\Omega_2 = \overline{2.8,10}$ at $x_3 = 0$ for the Ge-Si waveguide.

Determination of External Harmonic Influence on the Self-Oscillating System

The research methodology relies on employing the method of invariant relationships (IR), wherein the original system is dynamically expanded. This expansion involves uncovering additional connections between components of the mathematical model, both known and unknown, along the trajectories of the extended system. In essence, the approach involves systematically broadening the scope of the initial system, identifying new relationships within the mathematical model, and understanding how these connections evolve over time within the expanded framework.

Problems of determining the components of a mathematical model

$$\dot{x} = f(x, a, u), x(0) = x_0 \in R^n,$$
$$y(t) = h(x(t, x_0)) \in R^k$$

Components of the mathematical model: $x \in R^n$- state; $a \in R^p$- parameters; $u \in R^m$= control/input; $y \in R^k$- output.

Acceptable information about the system is the output and values that depend only on the output

$$Y(t) = \{y(t), \int y(s)ds, \dot{\xi} = f(t, \xi, y(t)), \dots\}$$

Problems:

Observation $Y(t) \Longrightarrow x(t)$;
Identification $Y(t) \Longrightarrow a$;
Inverse system $Y(t) \Longrightarrow u(t)$.

Synthesis of invariant relations in control problems

Original system $\dot{x} = f(x), y = h(x).$

Extended system $\begin{cases} \dot{x} = f(x) \\ \dot{p} = f(p) + u(p, h(x)) \end{cases}$

Synthesis of controls $\exists\, u^*: \quad \forall\, \Psi(x,p) = 0$

1. $\Psi(x,p) = 0$ describes the invariant manifold of the extended system;
2. $rank \frac{\partial(h(x),\Psi(p,z))}{\partial x} = k + m;$
3. The manifold is globally attractive for all trajectories of the extended system.

$\Psi(x,p) = 0$ — additional relations for the state of the system

The invariant relations synthesis method

The idea is to submerge the dynamics of the original system into a higher − dimensional system which, thanks to the controllable structure, is adapted to construct a manifold M on which the unknown values of mathematical model are functions of the known output

Determination of external harmonic effect on an oscillating system

Nonlinear oscillator $\ddot{x}(t) + f\big(x(t)\big)\dot{x}(t) + g(x(t)) = u(t).$

External harmonic impact $u(t) = a_1 + a_2 \cos \omega t + a_3 \sin \omega t.$

The method of IR is used to solve the problem of parametric identification: to find asymptotically accurate estimates of the coefficients $a_i, i = 1,2,3$ based on the output data $x_1(t)$, $x_2(t)$.

Derivation of invariant relations

Dynamic extension of the original model $\dot{\xi} = U\big(t, \xi, x_1(t), x_2(t)\big), \quad \xi(0) = \xi_0 \in R^3,$

Let us find additional relations on the trajectories of the extended system in the form $a_i = \Psi_i(t, x_1, x_2) + \xi_i, \quad i = 1,2,3.$

Deviations from invariant relations $\varepsilon_i = a_i - \Psi_i(t, x_1, x_2) - \xi_i$

Equations in deviations

$$\dot{\varepsilon}_i = -\Psi'_{i_{x_1}}\big(x_2 - F(x_1)\big) - \Psi'_{i_{x_2}}(X + \varepsilon_1 + \varepsilon_2 \cos \omega t + \varepsilon_3 \sin \omega t) - \Psi'_{i_t} - \dot{\xi}_i,$$

where $X = -g(x_1) + \Psi_1 + \xi_1 + (\Psi_2 + \xi_2) \cos \omega t + (\Psi_3 + \xi_3) \sin \omega t.$

At the first stage let us define the structure of the additional differential equations

$$\dot{\xi}_i = -{\Psi'}_{i_{x_1}}(x_2 - F) - {\Psi'}_{i_{x_2}} X - {\Psi'}_{i_t},$$

Then the equations for deviations become homogeneous:

$$\dot{\varepsilon}_i = -{\Psi'}_{i_{x_2}}(\varepsilon_1 + \varepsilon_2 \cos \omega t + \varepsilon_3 \sin \omega t).$$

Therefore the trivial solution is $\varepsilon_1(t) = \varepsilon_2(t) = \varepsilon_3(t) \equiv 0.$

Stabilization of deviations

It is natural to choose the free functions $\Psi_i(t, x_1, x_2)$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$
Stabilization problem for non-autonomous equations in deviations

$\dot{\varepsilon}_i = -{\Psi'}_{i_{x_2}}(\varepsilon_1 + \varepsilon_2 x_3 + \varepsilon_3 x_4)$, $\dot{x}_3 = -\omega x_4, \dot{x}_4 = \omega x_3,$
$\Psi_1 = x_2 + \Phi_1, \Psi_2 = x_2 x_3(t) + \Phi_2, \Psi_3 = x_2 x_4(t) + \Phi_3$, where $\Phi_i(t, x_1)$ are arbitrary functions

Asymptotic estimates of unknowns

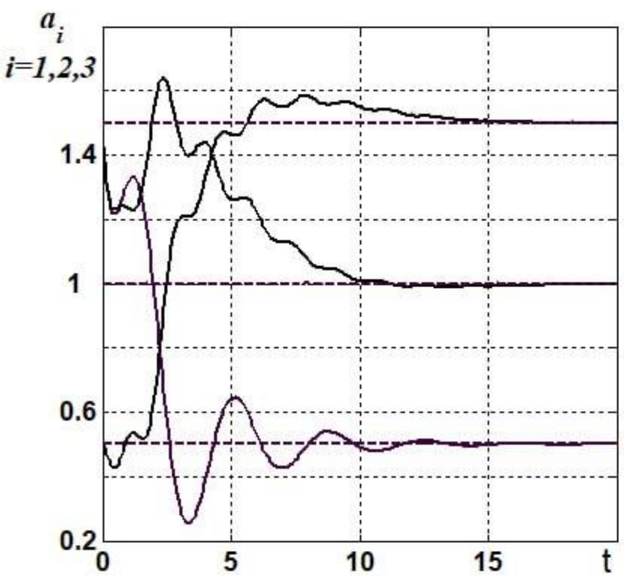
$$\hat{a}_1 = x_2(t) + \xi_1(t), \quad \hat{a}_2 = x_2(t) \cos \omega t + \xi_2(t), \quad \hat{a}_3 = x_2(t) \sin \omega t + \xi_3(t),$$

$\dot{\xi}_1 = B(t, x, \xi)$, $\quad \dot{\xi}_2 = B(t, x, \xi) \cos \omega t + \omega x_2 \sin \omega t$, $\quad \dot{\xi}_3 = B(t, x, \xi) \sin \omega t - \omega x_2 \cos \omega t$,
where $B(t, x, \xi) = g(x_1) - x_2 - \xi_1 - (x_2 \cos \omega t + \xi_2) \cos \omega t - (x_2 \sin \omega t + \xi_3) \sin \omega t$

Computational experiment. Duffing oscillator.

$$\ddot{x}(t) + \delta\dot{x}(t) + \alpha x(t) + \beta x^3(t) = a_1 + a_2 \cos \omega t + a_3 \sin \omega t.$$

$$B(t, x, \xi) = 4x_1 - 0.5x_1^3 - x_2 - \xi_1 - (x_2 \cos 2t + \xi_2) \cos 2t - (x_2 \sin 2t + \xi_3) \sin 2t.$$



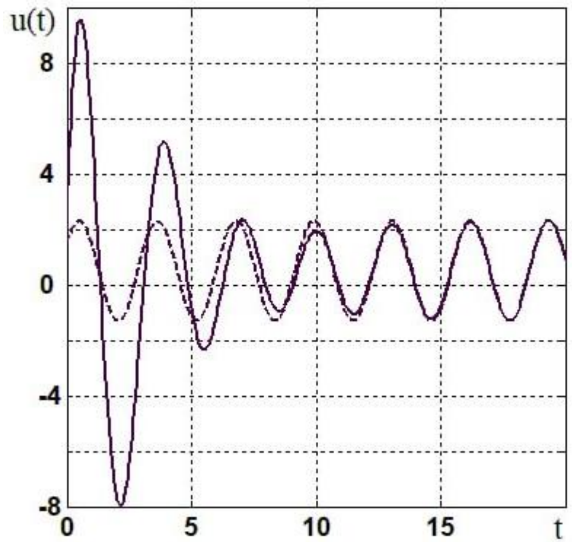


Fig.5 Asymptotic estimates of Duffing generator parameters and approximation of external signal

1. *Zhogoleva N.V.* Effects analysis of surface SH waves nonlinear interaction at the membrane covering of the waveguide surface / International science conf. “*Modern problems of mechanics and mathematics - 2023*” (May 23-25, 2023, Lviv, Ukraine).
   http://iapmm.lviv.ua/mpmm2023/materials/me04_02.pdf

2. *Zhogoleva N.V., Shcherbak V.F.* Determination of External Harmonic Influence on the Self-Oscillating System // Int. Appl. Mech.— 2023 — **59**, No. 2. –123-130.
   DOI 10.1007/s10778-023-01216-y