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STABILIZATION OF A SPINNING LAGRANGE GYROSCOPE FILLED WITH IDEAL FLUID IN A RESISTING MEDIUM

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1. Problem Statement. Basic Equations. Let dynamically symmetric rigid bodies S_1 and S_2 be connected at point O_2 by an elastic restoring spherical joint $\boldsymbol{L}_2 = k_2 \boldsymbol{s}_1 \times \boldsymbol{c}_2 / (|\boldsymbol{s}_1||\boldsymbol{c}_2|)$, $k_2 \geq 0$. The rigid body S_1 has an axially symmetric cavity τ that is fully filled with an ideal incompressible fluid of density ρ. The elastic restoring spherical joint $\boldsymbol{L}_1 = k_1 \nu \times \boldsymbol{s}_1 / |\boldsymbol{s}_1|$, $k_1 \geq 0$ acts on the body S_1 at a fixed point O_1. The rigid body S_i $(i = 1, 2)$ is under the force of gravity, dissipative torque $\boldsymbol{M}_i = \boldsymbol{D}_i \boldsymbol{\omega}_i$, $\boldsymbol{D}_i = \mathrm{diag}(D_{i1}, D_{i1}, D_{i3})$, $D_{ij} > 0$, $i = 1, 2$, $j = 1, 3$, which simulates a resisting medium, and a constant torque $\boldsymbol{M}_{iq} = Q_i e_3^i$ directed along the axis of symmetry of the rigid body S_i and sustains its uniform rotation. Here, $\boldsymbol{\omega}_i$ is the angular velocity of the rigid body S_i, $\boldsymbol{s}_1 = \overrightarrow{O_1 O_2}$ S_1, $\boldsymbol{c}_i = \overrightarrow{O_i C_i}$, m_i and C_i are the mass and the center of mass of the body S_i, $\boldsymbol{\nu} = -\boldsymbol{g}/|\boldsymbol{g}|$, $\boldsymbol{g}$ is the vector of acceleration of gravity, $g = |\boldsymbol{g}|$ (Fig. 1).

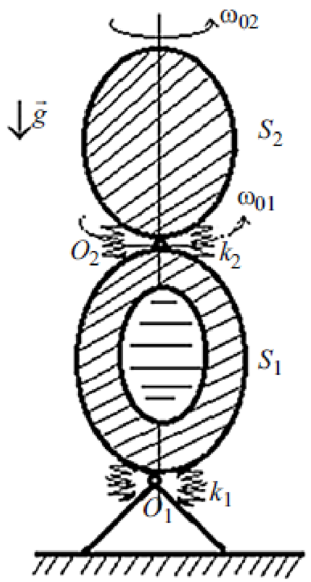

Fig. 1

$$(J_1 \cdot \boldsymbol{\omega}_1 + \boldsymbol{\lambda})^{\cdot} + m_2 \boldsymbol{s}_1 \times (\boldsymbol{\omega}_1 \times \boldsymbol{s}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{c}_2)^{\cdot} = (m_1 \boldsymbol{c}_1 + m_2 \boldsymbol{s}_1) g \times \boldsymbol{\nu} + \boldsymbol{L}_1 - \boldsymbol{L}_2 + \boldsymbol{M}_{1q} - D_1 \boldsymbol{\omega}_1,$$

$$(J_2 \cdot \omega_2)^{\cdot} + m_2 \boldsymbol{c}_2 \times (\omega_1 \times \boldsymbol{s}_1 + \omega_2 \times \boldsymbol{c}_2)^{\cdot} = m_2 \boldsymbol{c}_2 g \times \boldsymbol{\nu} + \boldsymbol{L}_2 + \boldsymbol{M}_{2q} - D_2 \omega_2. \qquad (1)$$

2. Stabilization Conditions for the Unstable Uniform Rotation of a Lagrange Gyroscope Filled with an Ideal Fluid

$$\dot{S}_n' + i(\omega_{01} - \lambda_n)S_n' + (\ddot{\gamma}_1 + i\omega_{01}\dot{\gamma}_1)e_n / N_n^2 = 0 \quad (n = 1, 2, \ldots) \qquad (2)$$

$$A_1'\ddot{\gamma}_1 + (i\tilde{C}_1 + D_{11})\dot{\gamma}_1 + \mu\ddot{\gamma}_2 + \sum_{n=1}^{\infty} e_n \dot{S}_n' = (a_1 g - k_1 - k_2)\gamma_1 \qquad (3)$$

$$A_2'\ddot{\gamma}_2 + (i\tilde{C}_2 + D_{21})\dot{\gamma}_2 + \mu\ddot{\gamma}_1 = (a_2 g - k_2)\gamma_2 \qquad (4)$$

$$\begin{vmatrix} F_1 & \mu\lambda^2 \\ \mu\lambda^2 & F_2 \end{vmatrix} = 0 \qquad (5)$$

where

$$F_1 = A_1'\lambda^2 + (i\tilde{C}_1 + D_1)\lambda - a_1 g + k_1 + k_2 - \lambda^2(i\lambda - \omega_{01})\sum_{n=1}^{\infty} \frac{E_n}{i\lambda - \tilde{\lambda}_n}$$

$$F_2 = A_2'\lambda^2 + (i\tilde{C}_2 + D_2)\lambda - a_2 g + k_2 \quad \tilde{\lambda}_n = (1 - \lambda_n)\omega_{01} \quad \lambda_n = 2/\kappa_n \quad E_n = \frac{2e_n^2}{N_n^2}$$

If n=1 then Eq. (5) becomes

$$a_5\lambda^5+\left(a_4+ib_4\right)\lambda^4+\left(a_3+ib_3\right)\lambda^3+\left(\tilde{a}_2+ib_2\right)\lambda^2+\left(\tilde{a}_1+ib_1\right)\lambda+a_0+ib_0=0 \qquad (6)$$

Here:

$$\begin{aligned}
a_5 &= A_1^* A_2' - \mu^2 = \left(A_1 - E_1\right)A_2 + \tilde{A}_2 s_1^2 m_2 > 0,\; b_5 = 0\\
a_4 &= A_1^* D_2 + A_2' D_1,\; b_4 = \tilde{C}_1^* A_2' + \tilde{C}_2 A_1^* + (A_1' A_2' - \mu^2)\tilde{\lambda}_1\\
a_3 &= D_1 D_2 - \tilde{C}_1 \tilde{C}_2 + \left(k_2 - a_2 g\right)A_1' + \left(k_1 + k_2 - a_1 g\right)A_2' - (\tilde{C}_1 A_2' + \tilde{C}_2 A_1')\tilde{\lambda}_1,\\
b_3 &= \tilde{C}_2 D_1 + C_1^* D_2 + (A_1' D_2 + A_2' D_1)\tilde{\lambda}_1,\\
\tilde{a}_2 &= \left(k_2 - a_2 g\right)D_1 + \left(k_1 + k_2 - a_1 g\right)D_2 - (\tilde{C}_2 D_1 + \tilde{C}_1 D_2)\tilde{\lambda}_1,\\
b_2 &= \tilde{C}_1^*\left(k_1 + k_2 - a_1 g\right) + \tilde{C}_2\left(k_2 - a_2 g\right) + \left[D_1 D_2 - \tilde{C}_1\tilde{C}_2 + A_1'\left(k_2 - a_2 g\right) + A_2'\left(k_1 + k_2 - a_1 g\right)\right]\tilde{\lambda}_1,\\
\tilde{a}_1 &= \left(k_1 + k_2 - a_1 g\right)\left(k_2 - a_2 g\right) - \left[\left(k_2 - a_2 g\right)\tilde{C}_1 + \left(k_1 + k_2 - a_1 g\right)\tilde{C}_2\right]\tilde{\lambda}_1,\\
b_1 &= \left[\left(k_2 - a_2 g\right)D_1 + \left(k_1 + k_2 - a_1 g\right)D_2\right]\tilde{\lambda}_1,\\
a_0 &= 0,\; b_0 = \left(k_1 + k_2 - a_1 g\right)\left(k_2 - a_2 g\right)\tilde{\lambda}_1.
\end{aligned} \qquad (7)$$

For all the zeros of Eq. (6) to be on the open left half-plane, according to the Lienard–Chipart criterion in inner form, it is necessary and sufficient that the ninth-order matrix composed of the coefficients of polynomial (6) be inner-positive, that is, the matrices $\Delta_1, \Delta_3, \Delta_5, \Delta_7$ and Δ_9 be positive definite:

$$I_1 = |\Delta_1| = a_4 = A_1^* D_2 + A_2' D_1 > 0 \tag{8}$$

$$I_3 = |\Delta_3| = \begin{vmatrix} a_5 & -b_4 & -a_3 \\ 0 & a_4 & -b_3 \\ a_4 & -b_3 & -\tilde{a}_2 \end{vmatrix} > 0 \tag{9}$$

$$I_5 = |\Delta_5| = \begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 \\ 0 & a_5 & -b_4 & -a_3 & b_2 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \end{vmatrix} > 0 \tag{10}$$

$$I_7 = |\Delta_7| = \begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 \\ 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 \\ 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 \\ 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 \end{vmatrix} > 0 \qquad (11)$$

$$I_9 = |\Delta_9| = \begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 & 0 & 0 \\ 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 & 0 \\ 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 \\ 0 & 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 \\ 0 & 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \\ 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 & 0 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 & 0 & 0 \end{vmatrix} = b_0 \tilde{I}_9 > 0 \qquad (12)$$

Since $I_1 > 0$, the asymptotic stability of uniform rotations in a resisting medium of two Lagrange gyroscopes, one filled with a fluid, is determined by four inequalities. (9)-(12)

The stabilization conditions for the kinetic momentum $\tilde{C}_2 = C_2\omega_{02}$ has the form:

$$I_{32}\tilde{C}_2^2 + I_{31}\tilde{C}_2 + I_{30} > 0$$

$$I_{54}\tilde{C}_2^4 + I_{53}\tilde{C}_2^3 + \ldots + I_{51}\tilde{C}_2 + I_{50} > 0 \qquad (13)$$

$$I_{76}\tilde{C}_2^6 + I_{75}\tilde{C}_2^5 + \ldots + I_{71}\tilde{C}_2 + I_{70} > 0$$

$$\left(\tilde{I}_{96}\tilde{C}_2^6 + \tilde{I}_{95}\tilde{C}_2^5 + \ldots + \tilde{I}_{91}\tilde{C}_2 + \tilde{I}_{90}\right)\left(k_1 - a_1 g\right)\left(k_2 - a_2 g\right) > 0.$$

Here:

$$I_{32} = D_1 g_3 \quad I_{54} = I_{32} D_1\left(f_5 + h_5\right) \quad I_{76} = I_{32} D_1^2 \tilde{\lambda}_1^2 f_5 h_5$$

$$I_{96} = I_{32} D_1^2 \tilde{\lambda}_1^4 f_5^2 h_5 \quad g_3 = A_1^{*2} D_2 + \mu^2 D_1 > 0 \qquad (14)$$

$$f_5 = k_1 + k_2 - a_1 g; \; h_5 = E_1 \tilde{\lambda}_1\left(\tilde{\lambda}_1 - \omega_{01}\right) = E_1 \lambda_1 \omega_{01}^2\left(\lambda_1 - 1\right)$$

Let us find the conditions for the elasticity coefficients k_1 and k_2 and the eigenvalues λ_1 at which inequalities (13) hold at a rather high angular velocity ω_{02}.

From (14) it follows that the coefficients I_{54}, I_{76}, I_{96} are positive if

$$\begin{aligned} &k_1 + k_2 > a_1 g \\ &\lambda_1 > 1 \\ &k_2 > a_2 g \end{aligned} \qquad (15)$$

Hence, if inequalities (15) hold, the greatest coefficients in the system of inequalities (13) are positive, and it is possible to stabilize the unstable rotation of the original Lagrange gyroscope filled with ideal fluid if the angular velocity is rather high ω_{02}

In the case of an ellipsoidal cavity, we have $\lambda_1 = 2/(1+\beta^2)$, Where a, c are its semi-axes, and c is the semi-axis directed along the axis of rotation. In this case, it follows from inequality (15) that $c < a$ and the ellipsoidal cavity is compressed along the axis of rotation.

3. *Conclusion*

The possibility of stabilization of unstable uniform rotation in a medium-high with resistance of a "sleeping" Lagrangian gyroscope with an ideal fluid by means of a second rotating gyroscope is considered. It is shown that stabilization will be impossible in the absence of elasticity in the general joint and the coincidence of the center of mass of the second gyroscope with this joint.

On the basis of the Ljenar-Schipar criterion written in innor form, stabilization conditions are obtained in the form of a system of four inequalities, corresponding to the kinetic moment of the second gyroscope. Conditions were found for the coefficients of elasticity of hinges and the first tone of oscillation of an ideal fluid under which the older coefficients of these inequalities are positive. Using the example of a compressed ellipsoidal cavity, it is shown that stabilization will always be possible with a sufficiently large angular velocity of rotation of the second gyroscope. A comparison of cases of presence and absence of dissipation was made. It is shown that in the absence of dissipation, the ellipsoidal cavity can be not only compressed, but also stretched. This is the fundamental difference in these two cases.

Publications

1. *Kononov Yu.M., Sviatenko Ya.I.* Stabilization of a Spinning Lagrange Gyroscope Filled with Ideal Fluid in a Resisting Medium // *Int. Appl. Mech.* – 2023. – **59**, № 2. – P. 207 – 217. DOI 10.1007/s10778-023-01213-1.
2. *Kononov Yu.M., Sviatenko Ya.I.* On the stability of rotation in an environment with resistance of a suspended Lagrange gyroscope / International conference *"Mathematic problems of the technical mechanic"*. Annual scientific conference. MPTM 2023. Kyiv, Dnipro, Kamianske, Ukraine. Book of Abstracts (Part 1, April 18-20, 2023).– P.51–52 .
3. *Kononov Yu.M., Sviatenko Ya.I.* On the stability of rotation in an environment with resistance of a suspended Lagrange gyroscope under the action of a constant moment / International of science conf. *"Modern problems of mechanics and mathematics - 2023"* (May 23-25, 2023, Lviv, Ukraine). http://iapmm.lviv.ua/mpmm2023/materials/proceedings.mpmm2023.pdf.

Thank you for your attention