PROBLEMS OF DETERMINING THE COMPONENTS OF A MATHEMATICAL MODEL FOR "INPUT-OUTPUT" SYSTEMS

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Supported by the Grant EFDS-FL2-08 of the found The European Federation of Academies of Sciences and Humanities (ALLEA)

- Invariant relations method
- Development of methods for determining the state and identification of network parameters of nonlinear oscillators
- Determination of external harmonic influence on the self-oscillating system
- Determination of external influence on the self-oscillating system

Problems of determining the components of a mathematical model

$\dot{x} = f(x, a, u),\ x(0) = x_0 \in \mathbb{R}^n$

$y(t) = h\big(x(t, x_0)\big) \in \mathbb{R}^k$

Components of the mathematical model:

$x \in \mathbb{R}^n$ – state; $a \in \mathbb{R}^p$ – parameters; $u \in \mathbb{R}^m$ – control/input; $y \in \mathbb{R}^k$ – output

Acceptable information about the system is output and values that depend only on the output

$$Y(t) = \left\{ y(t), \int y(s)ds\,, \dot{\xi} = f\big(t, \xi, y(t)\big), \dots \right\}$$

Problems

Observation $Y(t) \Rightarrow x(t)$

Identification $Y(t) \Rightarrow a$

Inverse system $Y(t) \Rightarrow u(t)$

The research methodology is based on the method of invariant relationships (IR) by dynamically expanding of the original system and determining additional connections between known and unknown components of the mathematical model on the trajectories of the extended system

Synthesis of invariant relations in control problems

Original system $\dot{x} = f(x), \quad y = h(x)$

Extended system $\begin{cases} \dot{x} = f(x) \\ \dot{p} = f(p) + u\big(p, h(x)\big) \end{cases}$

Synthesis of controls $\exists\, u^* : \forall\, \Psi(x, p) = 0$

1) $\Psi(x, p) = 0$ describes the invariant manifold of the extended system;

2) $rank\, \frac{\partial\left(h(x), \Psi(p,z)\right)}{\partial x} = k + m;$

3) the manifold is globally attractive for all trajectories of the extended system.

$$\Downarrow$$

$\Psi(x, p) = 0$ – additional relations for the state of the system

Method for synthesis of invariant relations

The idea is to submerge the dynamics of the original system into a higher-dimensional system which, thanks to the controllable structure, is adapted to construct a manifold M on which the unknown values of mathematical model are functions of the known output

Original system $\dot{x} = f(x), \qquad y = h(x)$

Extended system $\begin{cases} \dot{x} = f(x) \\ \dot{p} = f(p) + u\big(p, h(x)\big) \end{cases}$

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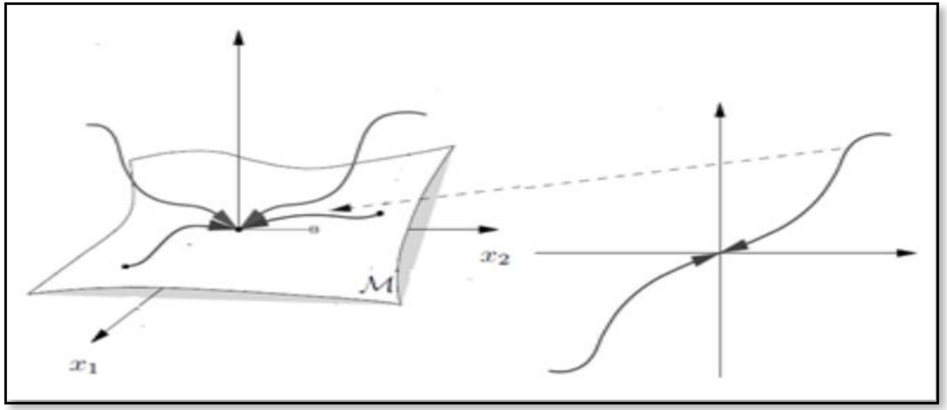


$\Psi(x, p) = 0$ describes the invariant manifold of the extended system;

Development of methods for determining the state and identification of network parameters of nonlinear oscillators

Balthasar van der Pol (1889-1959)

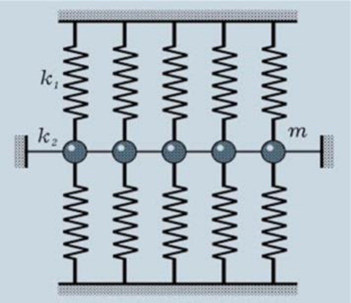
$$\dot{x} = v$$

$$\dot{v} = \boxed{g(\lambda - x^2)v} + \boxed{\omega^2} x$$

Dissipation/
Adding energy

Natural frequency

An oscillator with energy dissipation and accumulation is one of the natural fundamental building block of complex systems, inherent in biological organisms, neural networks, operating cycles of mechanisms, population dynamics, viral epidemics, etc.



Equations of motion of n interconnected oscillators

$$\ddot{x}_i + (\lambda_i - x_i^2)\dot{x}_i - \omega_i^2 x_i + \sum_{j(j\neq i)} \left[v_{ij}(x_i - x_j) + \mu_{ij}(\dot{x}_i - \dot{x}_j)\right], \qquad i,j = \overline{1,n}$$

Problems

By data about output $x_i(t)$, $\dot{x}_i(t)$ find asymptotically exact estimates of λ_i, ω_i, v_{ij}, μ_{ij}, $i,j = \overline{1,n}$

Development of methods for determining the state and identification of network parameters of nonlinear oscillators

Let us find additional relations on the trajectories of the extended system in the form

$$\lambda = \xi_1(t) - \Phi\big(x_1(t), x_2(t)\big) + \varepsilon_1$$

$$\omega^2 = \xi_2(t) - \Psi\big(x_1(t), x_2(t)\big) + \varepsilon_2$$

It is required to obtain differential equations for ξ and ε

$$\dot{\varepsilon}_1 = -\dot{\xi}_1 - \frac{\partial\Phi}{\partial x_1}x_2 - \frac{\partial\Phi}{\partial x_2}[(\varepsilon_1 + \xi_1 + \Phi - x_1^2)x_2 - (\varepsilon_2 + \xi_2 + \Psi)x_1],$$

$$\dot{\varepsilon}_2 = -\dot{\xi}_2 - \frac{\partial\Psi}{\partial x_1}x_2 - \frac{\partial\Psi}{\partial x_2}[(\varepsilon_1 + \xi_1 + \Phi - x_1^2)x_2 - (\varepsilon_2 + \xi_2 + \Psi)x_1],$$

$$\dot{\xi}_1 = -\frac{\partial\Phi}{\partial x_1}x_2 - \frac{\partial\Phi}{\partial x_2}[(\xi_1 + \Phi - x_1^2)x_2 - (\xi_2 + \Psi)x_1],$$

$$\dot{\xi}_1 = -\frac{\partial\Psi}{\partial x_1}x_2 - \frac{\partial\Psi}{\partial x_2}[(\xi_1 + \Phi - x_1^2)x_2 - (\xi_2 + \Psi)x_1],$$

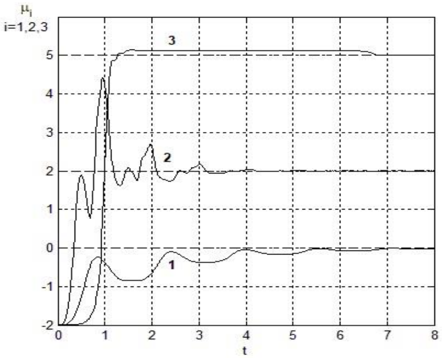
$$\dot{\varepsilon}_1 = -\frac{\partial\Phi}{\partial x_1}\varepsilon_1 x_2 - \frac{\partial\Phi}{\partial x_2}\varepsilon_2 x_1,$$

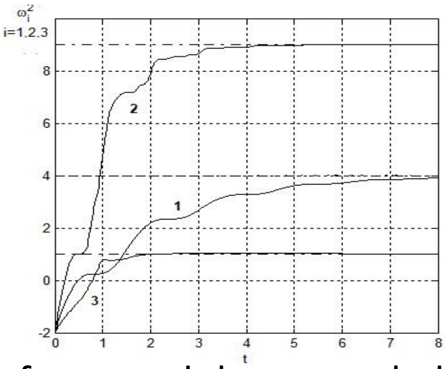
$$\dot{\varepsilon}_2 = -\frac{\partial\Psi}{\partial x_1}\varepsilon_1 x_2 - \frac{\partial\Psi}{\partial x_2}\varepsilon_2 x_1,$$

Final formulas for the asymptotic estimation of the characteristics of change dissipation (λ_i) for several van der Pol oscillators

$$\hat{\lambda}_i = \xi_i(t) + \frac{x_{2i}^2(t)}{2}, \qquad i = 1, 2, 3$$

$$\dot{\xi} = -x_{2i}\left[\left(\xi_i + \frac{x_{2i}^2}{2} - x_{2i-1}^2\right)x_{2i} - \omega_i^2 x_{2i-1} + F_i\right], \qquad \xi(0) \in \mathbb{R}^3$$





Asymptotic estimates of eigenvalues of two and three coupled van der Pol oscillators

Determination of external harmonic effect on an oscillating system

Nonlinear oscillator $\ddot{x}(t) + f\big(x(t)\big)\dot{x}(t) + g\big(x(t)\big) = u(t)$

External harmonic impact $u(t) = a_1 + a_2 \cos \omega t + a_3 \sin \omega t$

The method of IR is used to solve the problem of parametric identification: to find asymptotically accurate estimates of the coefficients a_i, $i = 1, 2, 3$, based on the output data $x_1(t)$, $x_2(t)$.

Derivation of Invariant Relations

Dynamic extension of the original model $\dot{\xi} = U\big(t, \xi, x_1(t), x_2(t)\big), \;\; \xi(0) = \xi_0 \in \mathbb{R}^3$

Let us find additional relations on the trajectories of the extended system in the form

$$a_i = \Psi_i(t, x_1, x_2) + \xi_i, \;\; i = \overline{1, 3}$$

Determination of external harmonic effect on an oscillating system

Deviations from invariant relations $$\varepsilon_i = a_i - \Psi_i(t, x_1, x_2) - \xi_i$$

Equations in deviations

$$\dot{\varepsilon}_i = -\Psi'_{i_{x_1}}\big(x_2 - F(x_1)\big) - \Psi'_{i_{x_2}}(X + \varepsilon_1 + \varepsilon_2 \cos\omega t + \varepsilon_3 \sin\omega t) - \Psi'_{i_t} - \dot{\xi}_i\,,$$

where $X = -g(x_1) + \Psi_1 + \xi_1 + (\Psi_2 + \xi_2)\cos\omega t + (\Psi_3 + \xi_3)\sin\omega t\,.$

As a first stage, let us define the structure of the additional differential equations

$$\dot{\xi}_i = -\Psi'_{i_{x_1}}(x_2 - F) - \Psi'_{i_{x_2}} X - \Psi'_{i_t}$$

Then the equations for deviations become homogeneous:

$$\dot{\varepsilon}_i = -\Psi'_{i_{x_2}}(\varepsilon_1 + \varepsilon_2 \cos\omega t + \varepsilon_3 \sin\omega t)$$

Therefore, there is the trivial solution $\varepsilon_1(t) = \varepsilon_2(t) = \varepsilon_3(t) = 0.$

Determination of external harmonic effect on an oscillating system

Stabilization of Deviations

It is natural to choose the free functions $\Psi_i(t, x_1, x_2)$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$

⇩

A stabilization problem for non-autonomous equations in deviations

$$\dot{\varepsilon}_i = -\Psi'_{i x_2}(\varepsilon_1 + \varepsilon_2 x_3 + \varepsilon_3 x_4),\ \dot{x}_3 = -\omega x_4,\ \dot{x}_4 = \omega x_3$$

Main tools: *Lyapunov functions; Barbashin-Krasovskii-LaSalle principle; Barbalat's lemma; property of persistently exciting*

$\Psi_1 = x_2 + \Phi_1, \Psi_2 = x_2 x_3(t) + \Phi_2, \Psi_3 = x_2 x_4(t) + \Phi_3, \Phi_i(t, x_1)$ are arbitrary functions

Final statement: Asymptotic estimates of unknowns

$\hat{a}_1 = x_2(t) + \xi_1(t), \hat{a}_2 = x_2(t)\cos\omega t + \xi_2(t), \hat{a}_3 = x_2(t)\sin\omega t + \xi_3(t),$

$\dot{\xi}_1 = B(t, x, \xi), \dot{\xi}_2 = B(t, x, \xi)\cos\omega t + \omega x_2 \sin\omega t, \dot{\xi}_3 = B(t, x, \xi)\sin\omega t - w x_2 \cos\omega t,$

where $B(t, x, \xi) = g(x_1) - x_2 - \xi_1 - (x_2\cos\omega t + \xi_2)\cos\omega t - (x_2 \sin\omega t + \xi_3)\sin\omega t$

Computational Experiment. Duffing Oscillator.

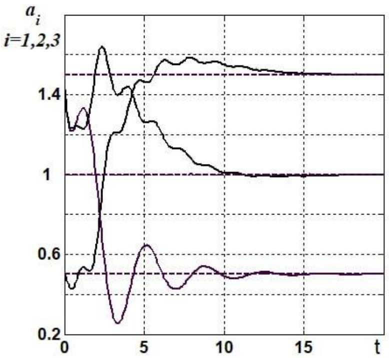
$$\ddot{x}(t) + \delta\dot{x}(t) + ax(t) + \beta x^3(t) = a_1 + a_2\cos\omega t + a_3\sin\omega t$$

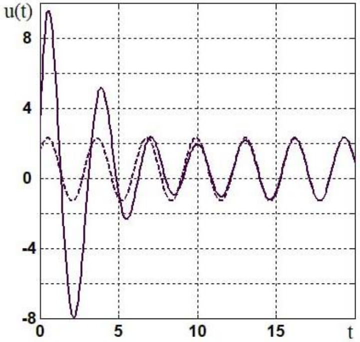
Estimates

$$\hat{a}_1 = x_2(t) + \xi_1(t), \hat{a}_2 = x_2(t)\cos\omega t + \xi_2(t), \hat{a}_3 = x_2(t)\sin\omega t + \xi_3(t),$$

$$\dot{\xi}_1 = B(t,x,\xi), \dot{\xi}_2 = B(t,x,\xi)\cos\omega t + \omega x_2\sin\omega t, \dot{\xi}_3 = B(t,x,\xi)\sin\omega t - wx_2\cos\omega t,$$

$$B(t,x,\xi) = 4x_1 - 0{,}5x_1^3 - x_2 - \xi_1 - (x_2\cos 2t + \xi_2)\cos 2t - (x_2\sin 2t + \xi_3)\sin 2t\ .$$





Asymptotic estimates of Duffing generator parameters and approximation of external signal

Determination of external influence on the self-oscillating system

Lienard oscillator $\ddot{s} + f(s)\dot{s} + g(s) = u; u(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n}[a_k \cos(kt) + b_k \sin(kt)]$

The goal is to determine the coefficients of the Fourier series segment in the form

$$\frac{1}{2}a_0 = \Phi_0(x_2) + \xi_0 + \varepsilon_0, \quad a_i = \Phi_i(x_2, t) + \xi_i + \varepsilon_i, \quad b_j = \Psi_j(x_2, t) + \xi_j + \delta_j, \quad j = \overline{1, n}$$

$$\dot{\varepsilon}_0 = -\acute{\Phi}_{0x2}(\varepsilon_0 + \varepsilon_1 \cos t + \delta_1 \sin t + \cdots + \varepsilon_n \cos nt + \delta_n \sin nt)$$

$$\dot{\varepsilon}_1 = -\acute{\Phi}_{1x2}(\varepsilon_0 + \varepsilon_1 \cos t + \delta_1 \sin t + \cdots + \varepsilon_n \cos nt + \delta_n \sin nt)$$

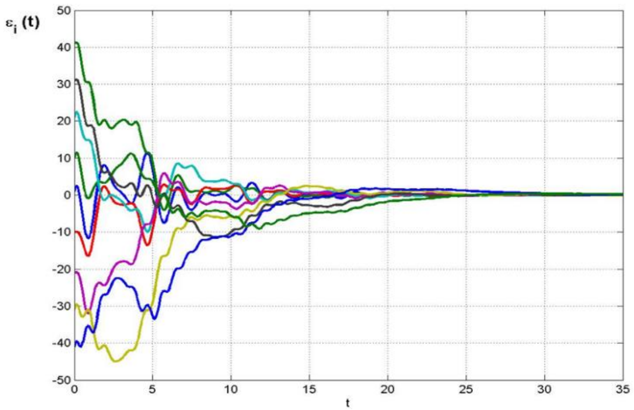
$$\dot{\delta}_n = -\acute{\Psi}_{nx2}(\varepsilon_0 + \varepsilon_1 \cos t + \delta_1 \sin t + \cdots + \varepsilon_n \cos nt + \delta_n \sin nt)$$

$$\Phi_0 = x_2, \quad \Phi_k = x_2 \cos kt, \quad \Psi_k = x_2 \sin kt, \quad k = \overline{1, n}$$

Determination of external influence on the self-oscillating system

$$\begin{pmatrix} \dot{\varepsilon}_0 \\ \dot{\varepsilon}_1 \\ \dot{\delta}_1 \\ \vdots \\ \dot{\varepsilon}_n \\ \dot{\delta}_n \end{pmatrix} = -\begin{pmatrix} 1 & cos(t) & sin(t) & cos(2t) & \cdots & sin(nt) \\ cos(t) & cos(t)\,cos(t) & cos(t)\,sin(t) & cos(t)\,cos(2t) & \cdots & cos(t)\,sin(nt) \\ sin(t) & sin(t)\,cos(t) & sin(t)\,sin(t) & sin(t)\,cos(2t) & \cdots & sin(t)\,sin(nt) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ cos(nt) & cos(nt)\,cos(t) & cos(nt)\,sin(t) & cos(nt)\,cos(2t) & \cdots & cos(nt)\,sin(nt) \\ sin(nt) & sin(nt)\,cos(t) & sin(nt)\,sin(t) & sin(nt)\,cos(t) & \cdots & sin(nt)\,sin(nt) \end{pmatrix} \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \delta_1 \\ \vdots \\ \varepsilon_n \\ \delta_n \end{pmatrix}$$

Equations in deviations have the *property of persistently exciting* hence its trivial solution is asymptotically stable



Error plots of 9 unknown coefficients in a model of an external force applied to a Duffing generator.

Publications

Shcherbak, V.F. (2023). Reconstruction of the Fourier expansion of a periodic force acting on an oscillator. *Journal of Mathematical Sciences*, 274(3), 383-391. https://doi.org/10.1007/s10958-023-06607-7

Zhogoleva, N.V. & Shcherbak, V.F. (2023). Determination of External Harmonic Effect on a Self-Oscillating System. *International Applied Mechanics*, 59(2), 238–244. https://doi.org/10.1007/s10778-023-01216-y

Thank you for attention!