# Generalized Green's operator of the matrix integral-differential boundary value problem unsolved with respect to the derivative

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# Structure of the talk

1. Statement of the Problem
2. Generalized Green's operator of the Cauchy problem
3. Generalized Green's operator of the integral-differential boundary value problem
4. Publications

# Statement of the Problem

We study the problem aimed at constructing the solutions

$$Z(t) \in \mathbb{D}^2[a; b], \quad Z'(t) \in \mathbb{L}^2[a; b], \quad Z(t) \in \mathbb{R}^{\alpha\times\beta}$$

of the matrix integral-differential system

$$\mathcal{D}Z(t) = \mathcal{I}Z(t) + F(t), \tag{1}$$

with the boundary condition

$$\mathcal{L}Z(\cdot) = \mathfrak{A}, \quad \mathfrak{A} \in \mathbb{R}^{\lambda\times\mu}. \tag{2}$$

# Statement of the Problem

Here, $\mathcal{D}$ is the differential operator

$$\mathcal{D}Z(t) := \sum_{i=1}^{p}\sum_{j=1}^{q} R_i(t)Z'(t)S_j(t) : \ \mathbb{D}^2[a;b] \to \mathbb{L}^2[a;b],$$

$\mathcal{I}$ is the integral operator

$$\mathcal{I}Z(t) := \Phi(t)\int_a^b \Big[A(s)Z(s) + B(s)Z'(s)\Big]\Psi(t)\,ds \ : \mathbb{D}^2[a;b] \to \mathbb{D}^2[a;b],$$

$\Phi(t) \in \mathbb{R}^{\gamma\times\eta}$, $A(t), B(t) \in \mathbb{R}^{\eta\times\alpha}$, $\Psi(t) \in \mathbb{R}^{\beta\times\delta}$, $R_i(t) \in \mathbb{R}^{\gamma\times\alpha}$, $S_i(t) \in \mathbb{R}^{\beta\times\delta}$ and $F(t) \in \mathbb{R}^{\gamma\times\delta}$ are matrices whose elements belong to the space $\mathbb{D}^2[a;b]$ of functions that are absolutely continuous in the interval $[a;b]$,
$\mathcal{L}Z(\cdot)$ is a linear bounded matrix functional,

$$\mathcal{L}Z(\cdot) : \ \mathbb{D}^2[a;b] \to \mathbb{R}^{\lambda\times\mu}.$$

Generally speaking, we assume $\alpha \neq \beta \neq \gamma \neq \delta \neq \eta \neq \lambda \neq \mu$.

# Previous studies


- *Boichuk A.A., Samoilenko A.M.* Generalized inverse operators and Fredholm boundary-value problems. Utrecht; Boston: VSP, 2004. XIV + 317 pp.
- *Chuiko S.M.* To the issue of a generalization of the matrix differential-algebraic boundary-value problem. *Journal of Mathematical Sciences,* 2017, **227**. Is. 1, pp. 13–25.
- *Boichuk A.A., Holovats'ka I.A.* Boundary-value problems for systems of integrodifferential equations. *Journal of Mathematical Sciences,* 2014, **203**, Is. 3, pp. 306–321.
- *Chuiko S.M., Chuiko O.V., Kuz'mina V.O.* Solutions to the boundary-value problem for a matrix integrodifferential equations with degenerate kernel. *Journal of Mathematical Sciences,* 2022, **263**, Is. 2, pp. 341–349.

# Generalized Green's operator of the Cauchy problem

Let us define the operator

$$\mathcal{A} = \mathcal{M}\mathcal{B} : \ \mathbb{R}^{m \times n} \to \mathbb{R}^{m \cdot n},$$

as an operator that associates the matrix $\mathcal{B} \in \mathbb{R}^{m \times n}$ with a column-vector $\mathcal{M}\mathcal{B} \in \mathbb{R}^{m \cdot n}$, composed of $n$ columns of the matrix $\mathcal{B}$, we also define the inverse operator

$$\mathcal{M}^{-1}\mathcal{A} : \ \mathbb{R}^{m \cdot n} \to \mathbb{R}^{m \times n},$$

which associates the vector $\mathcal{A} \in \mathbb{R}^{m \cdot n}$.
Denote the natural basis $\Theta^{(j)} \in \mathbb{R}^{\eta \times \beta}, \ j = 1, 2, \ \dots \beta \cdot \eta$ of the space $\mathbb{R}^{\eta \times \beta}$, and

$$\Xi^{(j)} \in \mathbb{R}^{\alpha \times \beta}, \ j = 1, 2, \ \dots \alpha \cdot \beta$$

denote the natural basis of the space $\mathbb{R}^{\alpha \times \beta}$.

# Generalized Green's operator of the Cauchy problem

The solution of the system (1) is sought in the form

$$Z(t) = \sum_{j=1}^{\alpha\cdot\beta} \Xi^{(j)} y_j(t), \quad y_j(t) \in \mathbb{R}^1, \quad y(t) \in \mathbb{R}^{\alpha\cdot\beta},$$

where the operator $\mathcal{M}\mathcal{D}Z(t) = \mathfrak{D}(t)y'(t) \ : \mathbb{R}^{\alpha\times\beta} \to \mathbb{R}^{\gamma\delta}$ is determined by the matrix

$$\mathfrak{D}(t) := \left\{\mathfrak{D}_i(t)\right\}_{k=1}^{\alpha\beta} \in \mathbb{R}^{\delta\gamma\times\alpha\beta}, \quad \mathfrak{D}_k(t) := \sum_{i=1}^{p}\sum_{j=1}^{q} R_i(t)\Xi^{(k)}S_j(t).$$

# Generalized Green's operator of the Cauchy problem

The operator $\mathcal{M}\mathcal{I}Z(t) = \mathcal{M}\Phi(t)C_0\Psi(t) \ : \mathbb{R}^{\alpha\times\beta} \to \mathbb{R}^{\gamma\delta}$ is determined by an unknown constant matrix

$$C_0 := \int_a^b \Big[A(s)Z(s) + B(s)Z'(s)\Big]\, ds \in \mathbb{R}^{\eta\times\beta},$$

$$C_0 = \sum_{j=1}^{\eta\cdot\beta} \Theta^{(j)}\xi_j, \quad \xi_j \in \mathbb{R}^1, \quad \xi \in \mathbb{R}^{\eta\cdot\beta},$$

In this case,

$$\mathcal{M}\mathcal{I}Z(t) = \Omega(t)\xi, \quad \Omega(t) := \Big\{\Omega_i(t)\Big\}_{i=1}^{\beta\eta} \in \mathbb{R}^{\delta\gamma\times\beta\eta},$$

$$\Omega_i(t) := \mathcal{M}\Phi(t)\Theta^{(i)}\Psi(t) \in \mathbb{R}^{\delta\gamma}.$$

# Generalized Green's operator of the Cauchy problem

The problem of constructing solutions of the Eq. (1) is equivalent to the standard differential-algebraic equation

$$\mathfrak{D}(t)y'(t) = \Omega(t)\xi + \mathcal{F}(t), \quad \mathcal{F}(t) := \mathcal{M}F(t).$$

Provided that the rank of the matrix $\mathfrak{D}(t)$ is complete and $P_{\mathfrak{D}^*(t)} = 0$, the latter problem leads to the ordinary differential equation

$$y'(t) = \mathcal{A}(t)\xi + \mathfrak{F}(t) + \psi(t), \tag{3}$$

whose solution

$$y(t) = \int_a^t \mathcal{A}(s)\xi \, ds + \int_a^t [\mathfrak{F}(s) + \psi(s)] \, ds + \zeta, \quad \zeta \in \mathbb{R}^{\alpha\cdot\beta}$$

determines the solution of the matrix Eq. (1), namely,

$$Z(t,\zeta,\xi) = \mathcal{M}^{-1}\int_a^t \mathcal{A}(s)\xi \, ds + \mathcal{M}^{-1}\int_a^t [\mathfrak{F}(s) + \psi(s)] \, ds + C_1,$$

$$C_1 := \mathcal{M}^{-1}[\zeta].$$

# Generalized Green's operator of the Cauchy problem

Here,

$$P_{\mathfrak{D}^*(t)} : \mathbb{R}^{\gamma\delta} \to \mathbb{N}(\mathfrak{D}^*(t)), \quad P_{\mathfrak{D}(t)} : \mathbb{R}^{\alpha\beta} \to \mathbb{N}(\mathfrak{D}(t))$$

are orthoprojector matrices,
$\varphi(t) \in \mathbb{D}^2[a; b]$ is an arbitrary vector function, and

$$\mathcal{A}(t) := \mathfrak{D}^+(t)\Omega(t), \quad \mathfrak{F}(t) := \mathfrak{D}^+(t)\mathcal{F}(t), \quad \psi(t) := P_{\mathfrak{D}(t)}\varphi(t) \in \mathbb{D}^2[a; b].$$

Provided that the rank of the matrix $\mathfrak{D}(t)$ is complete, the inclusion $\mathfrak{D}^+(t) \in \mathbb{D}^2[a; b]$ takes place.

Note that the ordinary differential equation (3) is equivalent to the matrix equation

$$Z'(t) = \mathcal{M}^{-1}\mathcal{A}(t)\xi + \mathcal{M}^{-1}\left[\mathfrak{F}(t) + \psi(t)\right].$$

# Generalized Green's operator of the Cauchy problem

Provided the completeness of the rank of the matrix $\mathfrak{D}(t)$ and under condition $P_{\Pi^*}\mathfrak{b} = 0$ the general solution

$$Z(t, c_\rho) = W(t, c_\rho) + \mathcal{K}\big[F(s); \varphi(s)\big](t), \quad c_\rho \in \mathbb{R}^\rho$$

of the Cauchy problem $Z(a) = C_1(\mathfrak{b}) + C_1(c_\rho)$ for system (1) determines the generalized Green's operator

$$\mathcal{K}\big[F(s); \varphi(s)\big](t) :=$$

$$:= \mathcal{M}^{-1}\int_a^t \mathcal{A}(s)\xi(\mathfrak{b})\, ds + \mathcal{M}^{-1}\int_a^t \left[\mathfrak{F}(s) + \psi(s)\right]\, ds + C_1(\mathfrak{b})$$

of the matrix Cauchy problem $Z(a) = C_1(\mathfrak{b})$ and

$$W(t, c_\rho) := C_1(c_\rho) + \mathcal{M}^{-1}\int_a^t \mathcal{A}(s)\xi(c_\rho)\, ds$$

is the general solution of the Cauchy problem $Z(a) = C_1(c_\rho)$ for the homogeneous part of Eq. (1).

# Generalized Green's operator for problem (1), (2)

Let

$$\Theta_\rho^{(j)} \in \mathbb{R}^\rho, \quad j = 1, 2, \ \ldots\ , \ \rho$$

denote the natural basis of the space $\mathbb{R}^\rho$. Substituting the solution of (1) into boundary condition (2), we get the problem of finding solutions

$$c_\rho = \sum_{j=1}^{\rho} \Theta_\rho^{(j)} \omega_j \in \mathbb{R}^\rho, \quad \omega_j \in \mathbb{R}^1, \quad j = 1, 2, \ \ldots\ , \ \rho$$

of the matrix equation of the Sylvester type

$$LW(\cdot, c_\rho) + L\mathcal{K}\big[F(s); \varphi(s)\big](\cdot) = \mathfrak{A} \in \mathbb{R}^{\mu \times \nu}. \tag{4}$$

Thus, the following sufficient condition for the solvability of the boundary-value problem for the matrix integral-differential system unsolved with respect to the derivative, is proved.

# Sufficient condition for the solvability

## Theorem

In the critical case ($P_{\mathcal{Q}^*} \neq 0$) under conditions

$$P_{\Pi^*}\mathfrak{b} = 0 \quad \text{and} \quad P_{\mathcal{Q}_d^*}\mathcal{M}\{\mathfrak{A} - L\mathcal{K}[F(s); \varphi(s)](\cdot)\} = 0,$$

and provided the completeness of the rank of the matrix $\mathfrak{D}(t)$, for an arbitrary vector function $\varphi(t) \in \mathbb{D}^2[a; b]$ the solution

$$Z(t, c_r) = W(t, c_r) + G[F(s); \varphi(s); \mathfrak{A}](t), \quad c_r \in \mathbb{R}^r$$

of the system (1), which satisfies the boundary condition (2) determines the generalized Green's operator

$$G[F(s); \varphi(s); \mathfrak{A}](t) :=$$
$$:= \mathcal{M}^{-1} \int_a^t \mathcal{A}(s)\xi(c_\rho(\mathcal{A}, F))\, ds + C_1(c_\rho(\mathcal{A}, F)) + \mathcal{K}[F(s); \varphi(s)](t)$$

# Sufficient condition for the solvability

and the general solution of the homogeneous part of the problem (1), (2), namely,

$$W(t, c_r) := C_1(c_\rho(c_r)) + \mathcal{M}^{-1} \int_a^t \mathcal{A}(s)\xi(c_\rho(c_r))\, ds;$$

here,

$$\begin{aligned} &\mathcal{K}\big[F(s); \varphi(s)\big](t) := \\ &\qquad := \mathcal{M}^{-1} \int_a^t \mathcal{A}(s)\xi(\mathfrak{b})\, ds + \mathcal{M}^{-1} \int_a^t \left[\mathfrak{F}(s) + \psi(s)\right]\, ds + C_1(\mathfrak{b}) \end{aligned}$$

is the generalized Green's operator of the Cauchy problem $Z(a) = C_1(\mathfrak{b})$ for the matrix integral-differential system (1).

# This results were published in...

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Thank you for your attention!