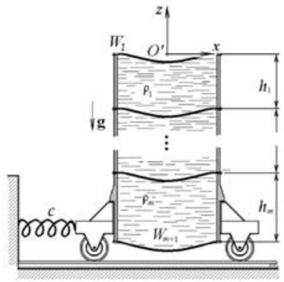
Yuriy Kononov

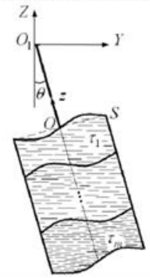
Institute of Applied Mathematics and Mechanics of the NAS of Ukraine

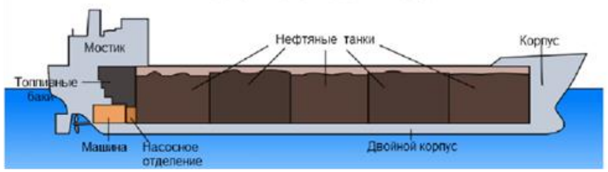
MATHEMATICAL MODELS OF THE MOTION OF A RIGID BODY WITH A LIQUID AND ELASTIC ELEMENTS

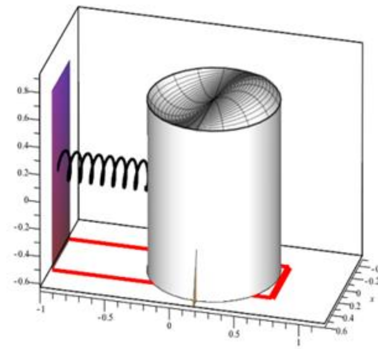
Supported by the Grant EFDS-FL2-08 of the found The European Federation of Academies of Sciences and Humanities (ALLEA)

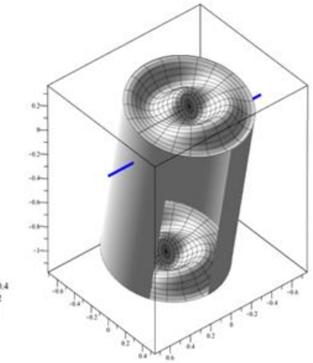
I. ON THE SOLUTION OF A COMPLICATED BIHARMONIC EQUATION IN A HYDROELASTICITY PROBLEM



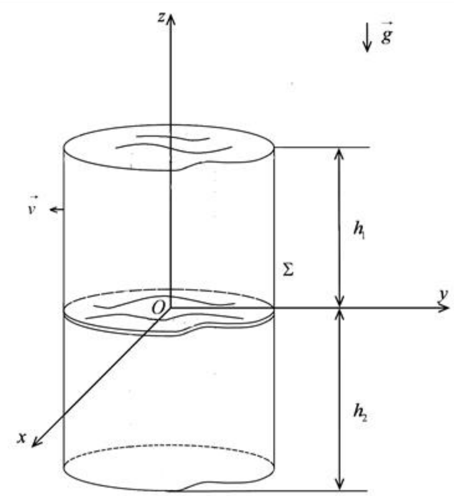








1. Boundary-value problem of common eigenmodes of the plate and the liquid



$$\Delta_2^2 w - 2P\Delta_2 w + q = \frac{\omega^2}{D}\sum_{n=1}^{\infty}\frac{a_n^* w_n}{k_n}\psi_n + C \quad (1)$$

$$w_n = \frac{1}{N_n^2}\int_S w\psi_n dS \quad (2)$$

$$\int_S w ds = 0 \quad (3)$$

$$\left(L_p[w]\right)\Big|_\gamma = 0 \quad (4)$$

Here $\left(\Delta_2\psi_n + k_n^2\psi_n\right)\Big|_S = 0, \ \frac{\partial \psi_n}{\partial \nu}\Big|_\gamma = 0$

Therefore, the eigenfrequencies and the forms of the common vibrations of the plate and the liquid are determined from the system of integro-differential equations (1) – (3) and boundary conditions (4). The obtained eigenfrequencies and eigen-functions are used in solving problems of the dynamics of the motion of a solid body with a liquid separated by elastic plates (see figures).

$w = \sum_{k=1}^{2} \left(w_k^0 - \tilde{w}_k^0 - \omega^2 \sum_{n=1}^{\infty} \frac{a_n^* E_{kn}^0}{\omega^2 a_n^* - k_n d_n} \psi_n \right) A_k^0$ is eigenfunctions.

On the example of a clamped plate, the eigenfrequencies equation was simplified by decomposing the corresponding homogeneous biharmonic equation into two harmonic equations and using Green's formula for the Laplace operator

$$\sum_{n=1}^{\infty} \frac{k_n}{\omega^2 a_n^* - k_n d_n} \left(\frac{B_n}{N_n} \right)^2 = 0$$

2.Some mechanical effects

For high frequencies $(n \gg 1)$, the following approximate formula is obtained $\omega_n^2 \approx \xi_{nm} \tilde{d}_{nm} / a \tilde{a}_{nm}$

where $\tilde{d}_{nm} = \left(D k_{nm}^2 + T \right) k_{nm}^2 + g \Delta \rho$

Approximate conditions of stability of free oscillations of the plate and liquid in a straight circular cylinder were obtained $D\left(\alpha_{1m} k_{2m}^4 + \alpha_{2m} k_{1m}^4\right) + T\left(\alpha_{1m} k_{2m}^2 + \alpha_{2m} k_{1m}^2\right) + g \Delta \rho \left(\alpha_{1m} + \alpha_{2m}\right) > 0$

This stability condition does not depend on the depth of liquid filling and the mass of the plate. It follows from this inequality that if the compressive forces or density of the upper liquid is greater than that of the lower one, joint oscillations of the plate and the liquid may become unstable. It also follows from this inequality that by pre-increasing the tension, the unstable oscillations of the plate and fluid can be stabilized.

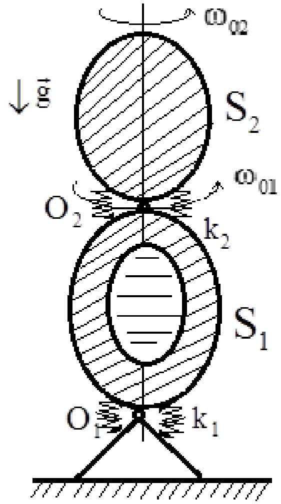
II. STABILIZATION OF A SPINNING LAGRANGE GYROSCOPE FILLED WITH IDEAL FLUID IN A RESISTING MEDIUM

This work continues the research that was started in the article (Kononov Y.N., Khomyak T.V. On the rotation stabilization of the unstable gyroscope containing fluid by rotating the rigid body // Facta Universitatis. Series Mechanics, Automatic Control and Robotics. – 2005. – 4, № 17. – P. 195 – 201) in case of no dissipation.

Based on the known equations of motion of the gyroscope system, P.V. Kharlamov and state functions for a rotating ideal fluid S.L. Sobolev, the problem of rotation in a medium with resistance of two Lagrangian gyroscopes, one of which has an ideal fluid, is considered.

The possibility of stabilizing unstable uniform rotation in a medium with resistance of a “sleeping” Lagrangian gyroscope with an ideal fluid using a second rotating gyroscope and elastic spherical joints is shown. The “sleeping” gyroscope has an arbitrary axisymmetric cavity completely filled with an ideal incompressible fluid and rotates around a fixed point, and a second gyroscope is located above it. Gyroscopes are connected by an elastic spherical joint, and their rotation is supported by constant moments directed along their axes of symmetry. The conditions for the elasticity coefficients of the hinges and the first tone of oscillation of an ideal fluid under which stabilization will be possible have been found. Using the example of a compressed ellipsoidal cavity, it is shown that at a sufficiently high angular velocity of rotation of the second gyroscope, stabilization will always be possible if the conditions for the elasticity coefficients are met.

The resulting stabilization conditions are compared with similar conditions in the absence of dissipation



1. *Problem Statement*

The equations of rotation in a resisting medium of two elastically coupled Lagrange gyroscopes, one having a cavity with an ideal fluid, has the following form:

$$(J_1 \cdot \boldsymbol{\omega}_1 + \boldsymbol{\lambda})^{\cdot} + m_2 \boldsymbol{s}_1 \times \left(\boldsymbol{\omega}_1 \times \boldsymbol{s}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{c}_2\right)^{\cdot} = \left(m_1 \boldsymbol{c}_1 + m_2 \boldsymbol{s}_1\right) g \times \boldsymbol{\nu} + \boldsymbol{L}_1 - \boldsymbol{L}_2 + \boldsymbol{M}_{1q} - D_1 \boldsymbol{\omega}_1$$

$$(J_2 \cdot \boldsymbol{\omega}_2)^{\cdot} + m_2 \boldsymbol{c}_2 \times \left(\boldsymbol{\omega}_1 \times \boldsymbol{s}_1 + \boldsymbol{\omega}_2 \times \boldsymbol{c}_2\right)^{\cdot} = m_2 \boldsymbol{c}_2 g \times \boldsymbol{\nu} + \boldsymbol{L}_2 + \boldsymbol{M}_{2q} - D_2 \boldsymbol{\omega}_2.$$

$\boldsymbol{\lambda} = \rho \int_\tau \mathbf{r} \times \mathbf{u} \, d\tau$ is the gyrostatic moment of the fluid

2. *Stabilization Conditions for the Unstable Uniform Rotation of a Lagrange Gyroscope Filled with an Ideal Fluid*

$$\dot{S}'_n + i(\omega_{01} - \lambda_n) S'_n + (\ddot{\gamma}_1 + i\omega_{01}\dot{\gamma}_1) e_n / N_n^2 = 0 \quad (n = 1, 2, \ldots) \qquad (1)$$

$$A_1'\ddot{\gamma}_1 + (i\tilde{C}_1 + D_{11})\dot{\gamma}_1 + \mu\ddot{\gamma}_2 + \sum_{n=1}^{\infty} e_n \dot{S}_n' = (a_1 g - k_1 - k_2)\gamma_1 \qquad (2)$$

$$A_2'\ddot{\gamma}_2 + (i\tilde{C}_2 + D_{21})\dot{\gamma}_2 + \mu\ddot{\gamma}_1 = (a_2 g - k_2)\gamma_2 \qquad (3)$$

$$\begin{vmatrix} F_1 & \mu\lambda^2 \\ \mu\lambda^2 & F_2 \end{vmatrix} = 0 \qquad (4)$$

$$F_1 = A_1'\lambda^2 + (i\tilde{C}_1 + D_1)\lambda - a_1 g + k_1 + k_2 - \lambda^2 (i\lambda - \omega_{01}) \sum_{n=1}^{\infty} \frac{E_n}{i\lambda - \tilde{\lambda}_n}; \;\; E_n = \frac{2e_n^2}{N_n^2};$$

$$F_2 = A_2'\lambda^2 + (i\tilde{C}_2 + D_2)\lambda - a_2 g + k_2$$

$$n = 1$$

$$a_5\lambda^5 + \left(a_4 + ib_4\right)\lambda^4 + \left(a_3 + ib_3\right)\lambda^3 + \left(\tilde{a}_2 + ib_2\right)\lambda^2 + \left(\tilde{a}_1 + ib_1\right)\lambda + a_0 + ib_0 = 0 \qquad (5)$$

From Eq. (4) it follows that if the center of mass of the second body coincides with the point, then this equation splits into two equations, with no influence of one body on another. Stabilization is impossible in this case.

For all the zeros of Eq. (5) to be on the open left half-plane, according to the Lienard–Chipart criterion in inner form, it is necessary and sufficient that the ninth-order matrix composed of the coefficients of polynomial (5) be inner-hositive, that is, the matrices $\Delta_1, \Delta_3, \Delta_5, \Delta_7$ and Δ_9 be positive definite:

$$I_1 = \left|\Delta_1\right| = a_4 = A_1^* D_2 + A_2' D_1 > 0$$

$$I_3 = \left|\Delta_3\right| = \begin{vmatrix} a_5 & -b_4 & -a_3 \\ 0 & a_4 & -b_3 \\ a_4 & -b_3 & -\tilde{a}_2 \end{vmatrix} > 0$$

$$I_5 = \left|\Delta_5\right| = \begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 \\ 0 & a_5 & -b_4 & -a_3 & b_2 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \end{vmatrix} > 0$$

$$I_7=\left|\Delta_7\right|=\begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 \\ 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 \\ 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 \\ 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 \end{vmatrix}>0$$

$$I_9=\left|\Delta_9\right|=\begin{vmatrix} a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 & 0 & 0 \\ 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 & 0 \\ 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 & 0 \\ 0 & 0 & 0 & a_5 & -b_4 & -a_3 & b_2 & \tilde{a}_1 & -b_0 \\ 0 & 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 \\ 0 & 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 \\ 0 & 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 \\ 0 & a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 & 0 \\ a_4 & -b_3 & -\tilde{a}_2 & b_1 & a_0 & 0 & 0 & 0 & 0 \end{vmatrix}=b_0\tilde{I}_9>0$$

Since $I_1 > 0$, he asymptotic stability of uniform rotations in a resisting medium of two Lagrange gyroscopes, one filled with a fluid, is determined by four inequalities. The stabilization conditions for the kinetic momentum $\tilde{C}_2 = C_2\omega_{02}$ has the form:

$$I_{32}\tilde{C}_2^2 + I_{31}\tilde{C}_2 + I_{30} > 0$$

$$I_{54}\tilde{C}_2^4 + I_{53}\tilde{C}_2^3 + ... + I_{51}\tilde{C}_2 + I_{50} > 0$$

$$I_{76}\tilde{C}_2^6 + I_{75}\tilde{C}_2^5 + ... + I_{71}\tilde{C}_2 + I_{70} > 0 \qquad (6)$$

$$\left(\tilde{I}_{96}\tilde{C}_2^6 + \tilde{I}_{95}\tilde{C}_2^5 + ... + \tilde{I}_{91}\tilde{C}_2 + \tilde{I}_{90}\right)\left(k_1 - a_1 g\right)\left(k_2 - a_2 g\right) > 0.$$

Here: $$I_{32} = D_1 g_3;\; I_{54} = I_{32} D_1 \left(f_5 + h_5\right)_5;\; I_{76} = I_{32} D_1^2 \tilde{\lambda}_1^2 f_5 h_5; \qquad (7)$$

$$I_{96} = I_{32} D_1^2 \tilde{\lambda}_1^4 f_5^2 h_5;\quad g_3 = A_1^{*2} D_2 + \mu^2 D_1 > 0;$$

$$f_5 = k_1 + k_2 - a_1 g;\; h_5 = E_1 \tilde{\lambda}_1 \left(\tilde{\lambda}_1 - \omega_{01}\right) = E_1 \lambda_1 \omega_{01}^2 \left(\lambda_1 - 1\right)$$

Let us find the conditions for the elasticity coefficients k_1 and k_2 and the eigenvalues λ_1 at which inequalities (6) hold at a rather high angular velocity ω_{02} .

From (7) it follows that the coefficients I_{54}, I_{76}, I_{96} are positive if

$$k_1 + k_2 > a_1 g$$

$$\lambda_1 > 1 \qquad (8)$$

$$k_2 > a_2 g$$

Hence, if inequalities (8) hold, the greatest coefficients in the system of inequalities (7) are positive, and it is possible to stabilize the unstable rotation of the original Lagrange gyroscope filled with ideal fluid if the angular velocity is rather high ω_{02}

In the case of an ellipsoidal cavity, we have $\lambda_1 = 2/(1+\beta^2)$, where a, c are its semi-axes, and c is the semi-axis directed along the axis of rotation. In this case, it follows from inequality (8) that $c < a$ and the ellipsoidal cavity is compressed along the axis of rotation.

3. *Conclusion*

The possibility of stabilization of unstable uniform rotation in a medium-high with resistance of a "sleeping" Lagrangian gyroscope with an ideal fluid by means of a second rotating gyroscope is considered. It is shown that stabilization will be impossible in the absence of elasticity in the general joint and the coincidence of the center of mass of the second gyroscope with this joint.

On the basis of the Ljenar-Schipar criterion written in innor form, stabilization conditions are obtained in the form of a system of four inequalities, corresponding to the kinetic moment of the second gyroscope. Conditions were found for the coefficients of elasticity of hinges and the first tone of oscillation of an ideal fluid under which the older coefficients of these inequalities are positive.

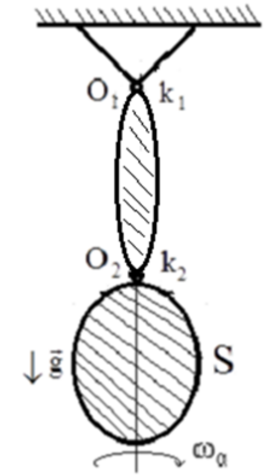
Using the example of a compressed ellipsoidal cavity, it is shown that stabilization will always be possible with a sufficiently large angular velocity of rotation of the second gyroscope. A comparison of cases of presence and absence of dissipation was made. It is shown that in the absence of dissipation, the ellipsoidal cavity can be not only compressed, but also stretched. This is the fundamental difference in these two cases.

III. ON THE STABILITY OF ROTATION IN AN ENVIRONMENT WITH RESISTANCE OF A SUSPENDED LAGRANGE GYROSCOPE

The problem of the rotation of a rigid body on a suspension is a further generalization of the very important and classical problem of the rotation of a rigid body around a fixed point. The string suspension was used to experimentally determine the direction of the main axes of inertia of complex mechanical structures, to balance turbines and heavy centrifuges, and in many other related tasks.

The considered problem is the stability of uniform rotation in a medium with resistance of a Lagrangian gyroscope on a suspension.

The stability conditions with respect to the kinetic moment will be written as follows:



$$I_{32}\tilde{C}_2^2 + I_{31}\tilde{C}_2 + I_{30} > 0,\ I_{56}\tilde{C}_2^6 + I_{55}\tilde{C}_2^5 + \ldots + I_{51}\tilde{C}_2 + I_{50} > 0$$

$$I_{56}\tilde{C}_2^6 + I_{55}\tilde{C}_2^5 + \ldots + I_{51}\tilde{C}_2 + I_{50} > 0 \qquad (1)$$

$$I_{78}\tilde{C}_2^8 + I_{77}\tilde{C}_2^7 + \ldots + I_{71}\tilde{C}_2 + I_{70} > 0$$

Analytical studies of the obtained stability conditions (1) from the main mechanical parameters of the system were carried out. Thus, in the case of degeneracy of the suspension into a massless and inertialess string, the conditions of stability (1) in dimensionless variables have the form:

$$I\xi - ma(a+b) < 0,\ I\xi^2 - \left[I + ma(a+b)\right] + mab > 0 \qquad (2)$$

It should be noted that the first inequality is absent in the article (*Karapetyan A.V., Lagutyna I.S.* On the stability of a uniform rotating top suspended on a string, taking into account the dissipative and constant moments // Izv. RAS. Rigid body mechanics. – 2000. – №. 1. – P. 53 – 57), and the second inequality coincides.

References

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Thank you for your attention