A Dynamic Observer for a Class of Infinite-Dimensional Vibrating Flexible Structures

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Observation problem

Linear system with output

$$\dot{z} = Az + Bu, \qquad (\Sigma)$$
$$y = Cz$$

$z(t) \in H$ — state vector, $u(t)$ — control, $y(t)$ — output,

H — Hilbert space, $u(t)$ and $y(t)$ are finite-dimensional vectors.

Luenberger observer

$$\dot{\bar{z}}(t) = (A - FC)\bar{z}(t) + Bu(t) + Fy(t), \qquad (\Omega)$$

A, B, C are given operators.

Problem: how to construct F, such that

$\|z(t) - \bar{z}(t)\| \to 0$ as $t \to +\infty$?

Observation error: $e(t) = z(t) - \bar{z}(t)$.

Error dynamics: $\dot{e}(t) = (A - FC)e(t)$. $\qquad (\Theta)$

Observer convergence: $\|e(t)\| \to 0$ as $t \to \infty$.

Infinite-dimensional control system with output

$$(\Sigma) \qquad \begin{aligned} \dot z &= Az + Bu, \\ y &= Cz \end{aligned}$$

$z(t) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in H = \ell^2 \times \ell^2$ — state vector,

$\xi = (\xi_1, \xi_2, \dots)^T \in \ell^2$, $\eta = (\eta_1, \eta_2, \dots)^T \in \ell^2$,

$A : D(A) \to H$,

$$D(A) = \left\{ z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in H : \sum_{j=1}^{\infty} \omega_j^2 (\xi_j^2 + \eta_j^2) < \infty \right\},$$

$$A : z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto Az = \begin{pmatrix} \Omega\eta \\ -\Omega\xi \end{pmatrix},$$

$\Omega = \mathrm{diag}(\omega_1, \omega_2, \dots)$.

$u = (F_0, M_1, \dots, M_k)^T \in \mathbb{R}^{k+1}$ — control,

$B : \mathbb{R}^{k+1} \to H, \quad Bu = \begin{pmatrix} 0 \\ B_1 u \end{pmatrix}, \quad B_1 : \mathbb{R}^{k+1} \to \ell^2,$

$$B_1 = \begin{pmatrix} b_{10} & b_{11} & \dots & b_{1k} \\ b_{20} & b_{21} & \dots & b_{2k} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \sum_{j=1}^{\infty} b_{ji}^2 < \infty, \; i = 0, \dots, k,$$

$y \in \mathbb{R}^r$ — output,

$C : H \to \mathbb{R}^r$, $Cz = C_1 \xi$, $C_1 : \ell^2 \to \mathbb{R}^r$,

$$C_1 = \begin{pmatrix} c_{11} & c_{12} & \dots \\ \vdots & \vdots & \vdots \\ c_{r1} & c_{r2} & \dots \end{pmatrix}, \quad \sum_{j=1}^{\infty} c_{sj}^2 < \infty, \; s = 1, \dots, r.$$

Assumption 1

The diagonal entries ω_j of Ω are mutually distinct positive real numbers such that

$$(A1) \qquad 0 < \omega_1 < \omega_2 < \cdots < \omega_n < \dots .$$

Lemma 1. Semigroup generating property

The operator A generates the C_0-semigroup $\left\{e^{tA}\right\}_{t\geqslant 0}$ on H.

$$D(A) = \left\{ z = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in H : \sum_{j=1}^{\infty} \omega_j^2(\xi_j^2 + \eta_j^2) < \infty \right\} \Subset H = \ell^2 \times \ell^2 \quad \Rightarrow \quad A \text{ is densely-defined in } H;$$

$\langle Az, z\rangle_H \equiv 0$ for any $z \in D(A)$ $\Rightarrow$ A is dissipative;

inverse operator: $A^{-1} : H \to H$, $A^{-1} = \begin{pmatrix} 0 & -\tilde{A}_1 \\ \tilde{A}_1 & 0 \end{pmatrix}$, $\tilde{A}_1 = \mathrm{diag}\left(\frac{1}{\omega_1}, \frac{1}{\omega_2}, \dots\right)$;

A^{-1} is bounded: $\|A^{-1}z\|^2 \leq \frac{1}{\omega_1^2}\|z\|^2$, $\forall\, z \in H$ $\Rightarrow$ A is closed.

Resolvent: $R(\lambda; A) = (I - \lambda A)^{-1} : H \to H$,

$$R(\lambda; A) = \begin{pmatrix} R_1 & R_2 \\ -R_2 & R_1 \end{pmatrix},$$

$$R_1 = \mathrm{diag}\left(\frac{1}{\lambda^2\omega_j^2 + 1},\ j = 1, 2, \dots\right), \qquad R_2 = \mathrm{diag}\left(\frac{\lambda\omega_j}{\lambda^2\omega_j^2 + 1},\ j = 1, 2, \dots\right) \quad \text{for some} \quad \lambda > 0.$$

The resolvent of A is well-defined for any $z \in H$ $\Rightarrow$ A is maximal.

Lemma 1. Semigroup generating property

The operator A generates the C_0-semigroup $\left\{e^{tA}\right\}_{t\geqslant 0}$ on H.

The operator A is

- ✓ densely-defined in H,
- ✓ closed,
- ✓ m-dissipative.

Lumer–Phillips theorem[a]: A is infinitesimal generator of C_0-semigroup $\left\{e^{tA}\right\}_{t\geqslant 0}$ on H.

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[a] Lumer G., Phillips R. S. (1961). Dissipative operators in a Banach space. Pacific J. Math., 11, 679–698.

Luenberger observer design

(Σ)
$$\dot{z} = Az + Bu,$$
$$y = Cz,$$

(Ω)
$$\dot{\bar{z}}(t) = (A - FC)\bar{z}(t) + Bu(t) + Fy(t),$$

$$F = \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^r \to H, \qquad f = \begin{pmatrix} f_{11} & \dots & f_{1r} \\ f_{21} & \dots & f_{2r} \\ \vdots & \vdots & \vdots \end{pmatrix} : \mathbb{R}^r \to \ell^2, \qquad g = \begin{pmatrix} g_{11} & \dots & g_{1r} \\ g_{21} & \dots & g_{2r} \\ \vdots & \vdots & \vdots \end{pmatrix} : \mathbb{R}^r \to H,$$

(Ω_1)
$$f_{js} = \gamma_s c_{sj}, \quad g_{js} = 0, \quad s = \overline{1, r}, \quad j = 1, 2, \dots,$$

$\gamma_s > 0$ are gain parameters.

Observer convergence

For any initial conditions $z(0), \bar{z}(0) \in H$ and any admissible control $u : [0, +\infty) \to \mathbb{R}^{k+1}$, the corresponding solutions $z(t)$ and $\bar{z}(t)$ of (Σ) and (Ω) satisfy

$$\|z(t) - \bar{z}(t)\| \to 0 \quad \text{as } t \to +\infty.$$

Observation error dynamics

$$\text{Observation error:}\quad e(t) = z(t) - \bar{z}(t), \qquad e(t) = \begin{pmatrix} \Delta \\ \delta \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \end{pmatrix}, \qquad \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix}.$$

(Θ)

$$\text{Error dynamics:}\quad \dot{e}(t) = (A - FC)e(t).$$

$$\text{Observer convergence:}\quad \|e(t)\| \to 0 \quad \text{as} \quad t \to \infty.$$

Goal: $e(t) \equiv 0$ is asymptotically stable.

Stability of $e(t) \equiv 0$

$$\mathcal{W}(e) = \sum_{j=1}^{\infty} (\Delta_j^2 + \delta_j^2) \text{ — positive definite quadratic form on } H,$$

its time derivative along the trajectories of (Θ) with

(Ω_1)

$$f_{js} = \gamma_s c_{sj}, \quad g_{js} = 0, \quad s = \overline{1, r}, \quad j = 1, 2, \ldots, \quad \gamma_s > 0$$

$$\text{is}\quad \dot{\mathcal{W}}(e)\Big|_{(\Theta),(\Omega_1)} = -2 \sum_{s=1}^{r} \gamma_s \left(\sum_{j=1}^{\infty} c_{sj} \Delta_j \right)^2 \leq 0, \quad \text{for all} \begin{pmatrix} \Delta \\ \delta \end{pmatrix} \in D(A).$$

Stability follows from the Lyapunov direct method.

Assumption 2

(A2) The series $\sum_{i=1}^{\infty} \frac{1}{\omega_i^2}$ is convergent.

Resolvent

$$R(\lambda; A - FC) = \begin{pmatrix} R^1 & R^2 \\ R^3 & R^4 \end{pmatrix},$$

$$R^1_{ji} = \frac{\lambda}{\lambda^2+\omega_j^2} \left(\frac{\lambda}{\lambda^2+\omega_i^2} \sum_{s,p=1}^{r} \gamma_s M_{sp}^{-1} c_{sj} c_{pi} - \boldsymbol{\delta_{ji}} \right), \qquad R^3_{ji} = -\frac{\omega_j}{\lambda} R^1_{ji},$$

$$R^2_{ji} = \frac{1}{\lambda^2+\omega_j^2} \left(\frac{\lambda\omega_i}{\lambda^2+\omega_j^2} \sum_{s,p=1}^{r} \gamma_s M_{sp}^{-1} c_{sj} c_{pi} - \omega_j \boldsymbol{\delta_{ji}} \right), \qquad R^4_{ji} = -\frac{1}{\lambda^2+\omega_j^2} \left(\frac{\omega_j\omega_i}{\lambda^2+\omega_i^2} \sum_{s,p=1}^{r} \gamma_s M_{sp}^{-1} c_{sj} c_{pi} + \lambda \boldsymbol{\delta_{ji}} \right),$$

$$M = (M_{sp}) = \lambda\gamma_p \sum_{i=1}^{\infty} \frac{c_{si} c_{pi}}{\lambda^2+\omega_i^2} + \boldsymbol{\delta_{sp}}, \quad s, p = \overline{1, r}, \; M^{-1} = (M_{sp}^{-1})_{s,p=\overline{1,r}}; \qquad \|M^{-1}\| \leq 1 + \mathcal{O}(\lambda) \quad \text{as } \lambda \to 0.$$

$$\|R(\lambda; A - FC)\|^2 \leqslant 2 \sum_{j,i=1}^{\infty} \frac{1}{\omega_j^2} \left(\frac{1}{\omega_i^2} \left(\sum_{s,p=1}^{r} \gamma_s M_{sp}^{-1} c_{sj} c_{pi} \right)^2 + 2 \right) < \infty,$$

$R(\lambda; A - FC) : H \to H$ is a compact operator for small $\lambda > 0$.

Lemma 2. Precompactness of the trajectories

If (A2) holds, then each positive semitrajectory $\{e(t)\}_{t \geqslant 0}$ of $\dot{e}(t) = (A - FC)e(t)$ is precompact in H.

Assumption 3

$(A3)$ The only invariant subspace of $\mathrm{Ker}\, C$ under the action of the semigroup $\left\{e^{tA}\right\}_{t \geqslant 0}$ is the singleton $\{0\}$.

Assumption 4

$(A4)$ For each $j \in \mathbb{N}$, there exists an $s = \overline{1, r}$ such that $c_{sj} \neq 0$.

Attractivity of the trajectories

(Θ)
$$\dot{e}(t) = (A - FC)e(t).$$

$$S = \left\{e : \dot{\mathcal{W}}(e) \equiv 0\right\} = \mathrm{Ker}\, C,$$

$$\dot{\mathcal{W}}(e) = -2 \sum_{s=1}^{r} \gamma_s \left(\sum_{j=1}^{\infty} c_{sj} \Delta_j \right)^2 \equiv 0 \quad (\gamma_s > 0), \quad \Rightarrow \quad C\, e(t) \equiv 0.$$

Asymptotic stability follows from the Barbashin–Krasovskii–LaSalle theorem [a,b]

[a] Barbashin, E. A., Krasovskii, N. N. (1952). On the stability of motion in general. In Proceedings of the Academy of Sciences of the USSR, 86 (3), 453 – 456.
[b] LaSalle J. P. *Stability theory and invariance principles* // Dynamical systems. — 1976. — Vol. 1. — P. 211–222.

Theorem 1. Asymptotic stability

If

$(A1)$ $\quad 0 < \omega_1 < \omega_2 < \dots < \omega_n < \dots,$

$(A2)$ $\quad \sum\limits_{i=1}^{\infty} \frac{1}{\omega_i^2} < \infty,$

$(A3)$ $\quad$ The only invariant subspace of $\mathrm{Ker}\, C$ under the action of the semigroup $\left\{e^{tA}\right\}_{t\geqslant 0}$ is the singleton $\{0\}$,

$(A4)$ $\quad$ For each $j \in \mathbb{N}$, there exists an $s = \overline{1, r}$ such that $c_{sj} \neq 0$,

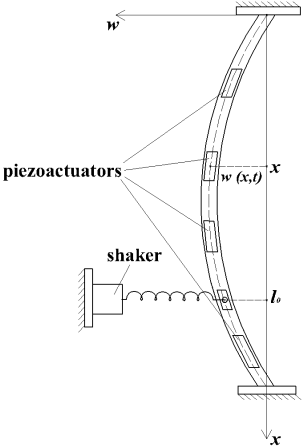
the components of the operator $F = \begin{pmatrix} f \\ g \end{pmatrix} : \mathbb{R}^r \to H$ are defined as $f = \begin{pmatrix} f_{11} & \dots & f_{1r} \\ f_{21} & \dots & f_{2r} \\ \vdots & \vdots & \vdots \end{pmatrix} : \mathbb{R}^r \to \ell^2$, $g = \begin{pmatrix} g_{11} & \dots & g_{1r} \\ g_{21} & \dots & g_{2r} \\ \vdots & \vdots & \vdots \end{pmatrix} : \mathbb{R}^r \to H$,

(Ω_1) $$f_{js} = \gamma_s c_{sj}, \quad g_{js} = 0, \quad \gamma_s > 0, \quad s = \overline{1, r}, \quad j = 1, 2, \dots,$$

then the trivial solution $e = 0$ of

(Θ) $$\dot{e}(t) = (A - FC)e(t).$$

is asymptotically stable.



Flexible structure oscillations

Euler–Bernoulli beam equation:

$$\rho \ddot{w}(x,t) + EIw''''(x,t) = \sum_{i=1}^{k} \psi_i''(x) M_j, \quad x \in (0,\ell) \setminus \{\ell_0\}, \tag{1}$$

$$w(x,t) \in C^2([0,\ell] \times [0,\tau]),\ \tau > 0;$$

$$w\Big|_{x=0} = w\Big|_{x=\ell} = 0, \quad w''\Big|_{x=0} = w''\Big|_{x=\ell} = 0, \tag{2}$$

$$(m\ddot{w} + \varkappa w)\Big|_{x=\ell_0} = EI\left(w'''\Big|_{x=\ell_0-0} - w'''\Big|_{x=\ell_0+0} \right) + F_0.$$

$EI > 0$, $\rho > 0$, $m > 0$, $\varkappa > 0$,

$\psi_j \in C^2(0,\ell)$, $\operatorname{supp} \psi_i \cap \{0, \ell_0, \ell\} = \emptyset$, $i = 1, 2, \ldots, k$,

$(F_0, M_1, \ldots, M_k)^T$ — control.

Mechanical parameters

$\ell = 1.875$ m — length of the beam, $\ell_0 = 1.4$ m,

$E = 7.1 \cdot 10^{10}$ Pa — Young modulus,

$I = 1.6875 \cdot 10^{-10}$ m^4 — cross-sectional inertia moment,

$\rho = 2660 \cdot 2.25 \cdot 10^{-4}$ kg/m — linear density,

$m = 0.045$ kg — mass of the shaker, $\varkappa = 2630$ N/m — stiffness of the spring,

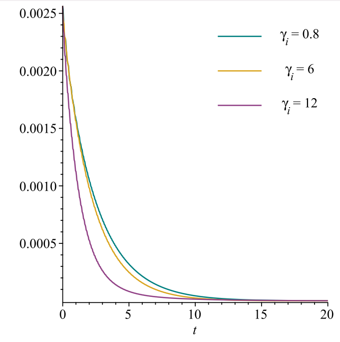
F_0 — force applied by the shaker at $x = \ell_0$,

$M_1, \ldots, M_k$ — torques of piezoactuators,

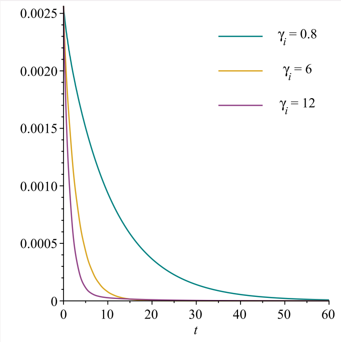
$\psi_1, \ldots, \psi_k$ — shape functions (distribution of actuators).

Numerical simulations

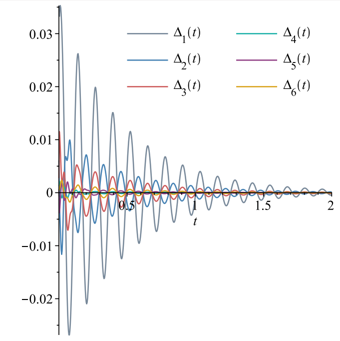
$$\|e_N(t)\|^2 = \frac{\varkappa}{2}\sum_{j=1}^{N}(\Delta_j(t))^2 + \frac{m}{2}\sum_{j=1}^{N}(\delta_j(t))^2$$



Time plot of $\|e_{16}\|^2$



Time plot of $\|e_{40}\|^2$



Time plot of $\Delta_1(t), \ldots, \Delta_6(t)$, $\gamma_i = 6$, $i = 1, \ldots, 6$.

Outcomes

- Luenberger-type observer design that allows to asymptotically reconstruct the full state of a diagonal system in the infinite-dimensional space operating by a finite number of outputs.
- Asymptotic stability of the error dynamics.
- The observer admits arbitrarily large inputs and unbounded outputs.
- For a particular flexible structure, the observer parameters are evaluated in terms of eigenvalues and eigenfunctions of the corresponding spectral problem.
- The estimation of the error decay rate is an open question.

Published paper

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THANK YOU!