Toward the theory of nonclassical solutions of equations of mathematical physics

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Absract of results

Our research was devoted to the study of the Beltrami equation with sources that is a complex form of one of the main equations of hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media, see e.g. Theorem 16.1.6 in the monograph Astala, K., Iwaniec, T., Martin, G.J. (2009) *Elliptic differential equations and quasiconformal mappings in the plane.* Princeton Math. Ser. **48**. Princeton: Princeton Univ. Press. First of all, we investigate existence and regularity of solutions to these equations.

Then we establish a series of effective integral criteria of the type of BMO, FMO, Calderon-Zygmund, Lehto and Orlicz on singularities of the equations at the boundary for existence, representation and regularity of its solutions. Moreover, we prove existence, representation and regularity of weak solutions of Dirichlet's problem for the corresponding Poisson type equations $\mathrm{div}[\mathrm{A}(\mathrm{z})\nabla\,\mathrm{u}(\mathrm{z})] = \mathrm{g}(\mathrm{z})$ with matrix valued coefficients $\mathrm{A}(\mathrm{z})$.

Absract of results

Finally, we obtain theorems on existence, representation and regularity of nonclassical solutions to the Beltrami equations with sources for the well-known Hilbert and Riemann problems whose coefficients and boundary dates are measurable with respect to logarithmic capacity under geometric interpretations of the boundary values in the sense of angular limits, along non-tangential paths, and along the Bagemihl-Seidel systems of arcs to the boundary.

Introduction

We first study the **Dirichlet problem** of the Bojarski-Vekua type

$$\lim_{z\to\zeta} \mathrm{Re}\,\omega(z) = \varphi(\zeta) \qquad \forall\, \zeta \in \partial D \tag{1}$$

with continuous boundary data $\varphi : \partial D \to \mathbb{R}$ in arbitrary bounded domains D in the complex plane $\mathbb{C}$ with no boundary component degenerated to a single point for the **inhomogeneous Beltrami equations**

$$\omega_{\bar z} \;=\; \mu(z)\cdot\omega_z \;+\; \sigma(z)\,, \quad z \in D\,, \tag{2}$$

with sources $\sigma : D \to \mathbb{C}$ in L_p, $p > 2$, where $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $\omega_{\bar z} = (\omega_x + i\omega_y)/2$, $\omega_z = (\omega_x - i\omega_y)/2$, $z = x + iy$, ω_x and ω_y are partial derivatives of the function ω in x and y, respectively.

Introduction

In such domains D, we also prove theorems on existence, representation and regularity of solutions for the **classical Dirichlet problem**

$$\lim_{z\to\zeta} u(z) = \varphi(\zeta) \quad \forall\,\zeta\in\partial D \tag{3}$$

with continuous boundary dates $\varphi : \partial D \to \mathbb{R}$ to the **Poisson type equations**

$$\mathrm{div}[A(z)\nabla\, u(z)] = g(z) \tag{4}$$

with sources $g : D \to \mathbb{R}$ in $L_p(D)$, $p > 1$.

Note that the corresponding examples show that the request on domains D to have no boundary component degenerated to a single point is necessary. Thus, our results on the Dirichlet problem (3) to the Poisson type equations (4) were obtained by us in the most general admissible domains.

Introduction

For the case $\|\mu\|_\infty < 1$, equation (2) was first introduced by L. Ahlfors and L. Bers (1960). We study the case of locally uniform ellipticity of the equation when its **dilatation quotient** K_μ is bounded only locally in D,

$$\mathrm{K}_\mu(\mathrm{z}) \ := \ \frac{1 \ + \ |\mu(\mathrm{z})|}{1 \ - \ |\mu(\mathrm{z})|} \ ,$$

i.e., if $\mathrm{K}_\mu \in \mathrm{L}_\infty$ on each compact set in D but admits singularities at the boundary. A point $\zeta \in \partial\mathrm{D}$ is called a **singular point of the equation** (2) if $\mathrm{K}_\mu \notin \mathrm{L}_\infty$ on each neighborhood of the point.

First of all, we show that, if D is an arbitrary simply connected domain in $\mathbb{C}$, then the Dirichlet problem (1) to the equation (2) has a solution ω in class $\mathrm{W}^{1,2}_{\mathrm{loc}}(\mathrm{D})$ for a wide circle of singularities of (2) at the boundary. Moreover, it is unique up to an additive pure imaginary constant, and it can be represented through the so-called generalized analytic functions with sources.

Introduction

Recall that the Vekua monograph was devoted to **generalized analytic functions**, i.e., continuous complex valued functions $\mathrm{H}(\mathrm{z})$ of one complex variable $\mathrm{z} = \mathrm{x} + \mathrm{iy}$ of class $\mathrm{W}^{1,1}_{\mathrm{loc}}$ in a domain D satisfying the equations

$$\partial_{\bar{\mathrm{z}}}\mathrm{H} \ + \ \mathrm{aH} \ + \ \mathrm{b}\overline{\mathrm{H}} \ = \ \mathrm{S}\ , \quad \partial_{\bar{\mathrm{z}}} \ := (\partial_{\mathrm{x}} + \mathrm{i}\partial_{\mathrm{y}})/2$$

with complex valued coefficients $\mathrm{a}, \mathrm{b}, \mathrm{S} \in \mathrm{L}_{\mathrm{p}}(\mathrm{D})$, $\mathrm{p} > 2$. A function H is called a **generalized analytic function with the source** S if $\mathrm{a} \equiv 0 \equiv \mathrm{b}$.

Then we establishe similar theorems on existence, representation and regularity of the so-called multi-valued solutions of the Dirichlet problem (1) with continuous boundary data $\varphi : \partial\mathrm{D} \to \mathbb{R}$ to the Beltrami equations with sources (2) in arbitrary domains D in $\mathbb{C}$ without any boundary components degenerated to a single point. We say that a function $\mathrm{h} : \mathrm{D} \to \mathbb{R}$ is a **weak generalized harmonic function with a source** S in D, if $\mathrm{h} = \mathrm{Re}\,\mathrm{H}$ for a generalized analytic function H with the source S.

The main lemma in simply connected domains

It is well known that the homogeneous Beltrami equation

$$f_{\bar{z}} = \mu(z) f_z \tag{5}$$

is the basic equation in analytic theory of quasiconformal and quasi-regular mappings in the plane with numerous applications in nonlinear elasticity, gas flow, hydrodynamics and other sections of natural sciences. Note that continuous functions f with generalized derivative by Sobolev $f_{\bar{z}} = 0$ are analytic functions that corresponds to the case $\mu(z) \equiv 0$ in (5).

The equation (5) is called **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$. It is known that if K_μ is bounded, then the equation has homeomorphic solutions in $W^{1,2}_{\mathrm{loc}}$ called **quasiconformal mappings**. Recently, a series of effective criteria for existence of homeomorphic solutions in $W^{1,1}_{\mathrm{loc}}$ have been also established for the degenerate Beltrami equations.

The main lemma in simply connected domains

These criteria were formulated both in terms of K_μ and the more refined quantity that takes into account not only the modulus of the complex coefficient μ but also its argument

$$K_\mu^T(z, z_0) \ := \ \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2}$$

called **tangent dilatation quotient** of Beltrami equations with respect to a point $z_0 \in \mathbb{C}$. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \qquad \forall\, z \in D\,,\ z_0 \in \mathbb{C}\,. \tag{6}$$

Let D be a domain in the complex plane $\mathbb{C}$. A function $f : D \to \mathbb{C}$ in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is called a **regular solution** of the Beltrami equation (5) if f satisfies (5) a.e. and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 > 0$ a.e. By V. Ryazanov, U. Srebro, E. Yakubov (2006) On the theory of the Beltrami equation. Ukr. Math. J. 58:11, we have the following statement on the existence of regular homeomorphic solutions f in $\mathbb{C}$ for the Beltrami equation (5).

The main lemma in simply connected domains

Proposition 1.

Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{1,loc}(\mathbb{C})$. Suppose that, for each $z_0 \in \mathbb{C}$ with $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0) \cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\, dm(z) \ = \ o(I^2_{z_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{7}$$

for a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\varepsilon_0) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, with

$$I_{z_0}(\varepsilon) : \ = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt \ < \ \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,. \tag{8}$$

Then the Beltrami equation (5) has a regular homeomorphic solution f^μ.

The main lemma in simply connected domains

Here $\mathrm{dm}(\mathrm{z})$ corresponds to the Lebesgue measure in $\mathbb{C}$ and by (6) $\mathrm{K}_\mu^\mathrm{T}$ can be replaced by K_μ. We call such solutions f^μ of (5) $\mu-$**conformal mappings**.

Lemma 1.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L}_\mathrm{p}(\mathrm{D})$, $\mathrm{p} > 2$, with compact support in D, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L}_1(\mathrm{D})$ and conditions (7) and (8) hold for all $\mathrm{z}_0 \in \partial\mathrm{D}$.

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $\mathrm{W}^{1,2}_{\mathrm{loc}}$ of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial\mathrm{D} \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

The main lemma in simply connected domains

Moreover, $\omega = h \circ f$, where $f : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D and $h : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S of the class $L_{p_*}(D_*)$ for some $p_* \in (2, p)$,

$$S \; := \; \sigma \cdot \frac{f_z}{J} \circ f^{-1} \; , \tag{9}$$

where $J = |f_z|^2 - |f_{\bar{z}}|^2$ is the Jacobian of f, that satisfies the Dirichlet condition

$$\lim_{w \to \zeta} \; \mathrm{Re}\, h(w) \; = \; \varphi_*(\zeta) \quad \forall \, \zeta \in \partial D_* \; , \qquad \text{where } \varphi_* := \varphi \circ f^{-1}|_{\partial D_*} .$$

We assume here and further that the dilatation quotients $K^T_\mu(z, z_0)$ and $K_\mu(z)$ are extended by 1 outside the domain D.

Remark 1.

In tern, the generalized analytic function h with the source S has the representation $\mathrm{h} = \mathcal{A} + \mathrm{H}$, where

$$\mathrm{H}(\mathrm{w}) \;=\; -\frac{1}{\pi}\int\limits_{\mathrm{D}_*} \frac{\mathrm{S}(\zeta)}{\zeta-\mathrm{w}}\,\mathrm{dm}(\zeta)\,, \quad \mathrm{w}\in\mathbb{C}\,,$$

with $\mathrm{H}_{\overline{\mathrm{w}}} = \mathrm{S}$, and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition

$$\lim_{\mathrm{w}\to\zeta}\ \mathrm{Re}\,\mathcal{A}(\mathrm{w}) \;=\; \varphi^*(\zeta) \quad \forall\,\zeta\in\partial\mathrm{D}_*\,, \quad \text{where } \varphi^* := \varphi_* - \mathrm{Re}\,\mathrm{H}|_{\partial\mathrm{D}_*}\,, \tag{10}$$

H is α_*-Hölder continuous in D_* with $\alpha_* = 1 - 2/\mathrm{p}_*$ and $\mathrm{H}|_{\mathrm{D}_*} \in \mathrm{W}^{1,\mathrm{p}_*}(\mathrm{D}_*)$. Note, f and f^{-1} are locally quasiconformal mappings in D and D_*, respectively.

Remark 2.

If a family of functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$, $z_0 \in \partial D$, in Lemma 1 is independent on ε, then condition (7) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Note also that (7) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0}^2(|z - z_0|)\, dm(z) < \infty \qquad \forall\, z_0 \in \partial D \tag{11}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the solvability of the Dirichlet problem (1) in D for the Beltrami equations with sources (2) for all continuous boundary functions φ, it is sufficient that the integral in (11) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z - z_0|)$, $\lambda \in (0, 1)$, $z \in \mathbb{D}$, $z_0 \in \partial\mathbb{D}$, and $\psi(t) = 1/(t\ \log(e/t))$, $t \in (0, 1)$, show that the condition (11) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Lemma 1 makes it to be possible to derive a number of effective integral criteria for solvability of the Dirichlet problem to the Beltrami equations with sources.

Recall first that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D, abbr. $u \in BMO(D)$, if $u \in L^1_{loc}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B| \, dm(z) < \infty \,, \tag{12}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int\limits_B u(z) \, dm(z) \,.$$

We write $u \in BMO_{loc}(D)$ if $u \in BMO(U)$ for each relatively compact subdomain U of D. We also write sometimes for short BMO and BMO_{loc}, respectively.

The main criteria in simply connected domains

The class BMO was introduced by John and Nirenberg (1961) and soon became an important concept in harmonic analysis, partial differential equations and related areas. A function φ in BMO is said to

have **vanishing mean oscillation**, abbr. $\varphi \in \mathrm{VMO}$, if the supremum in (12) taken over all balls B in D with $|\mathrm{B}| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in 1975. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO.

Remark 3.

Note that by Brezis-Nirenberg (1995) $W^{1,2}(D) \subset VMO(D)$.

Following Ignatev-Ryazanov (2005), we say that a function $\varphi : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in FMO(z_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} |\varphi(z) - \widetilde{\varphi}_\varepsilon(z_0)| \, dm(z) < \infty \,, \tag{13}$$

where

$$\widetilde{\varphi}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z) \, dm(z)$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0, \varepsilon) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Note that the condition (13) includes the assumption that φ is integrable in some neighborhood of the point z_0.

The main criteria in simply connected domains

Proposition 2.

If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi_\varepsilon| \, dm(z) < \infty \,,$$

then φ is of finite mean oscillation at z_0.

In particular, choosing here $\varphi_\varepsilon \equiv 0$, we obtain the following.

Corollary 1.

If, for a point $z_0 \in D$,

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} |\varphi(z)| \, dm(z) < \infty \,, \tag{14}$$

then φ has finite mean oscillation at z_0.

The main criteria in simply connected domains

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \to \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \fint_{B(z_0,\varepsilon)} |\varphi(z) - \varphi(z_0)| \, dm(z) = 0\,.$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L_1(D)$. Thus, we have by Proposition 2 the next corollary.

Corollary 2.

Every locally integrable function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at almost every point in D.

Remark 4.

Note that the function $\varphi(z) = \log\left(1/|z|\right)$ belongs to BMO in the unit disk Δ, and hence also to FMO. However, $\widetilde{\varphi}_{\varepsilon}(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (14) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0. Clearly, $\mathrm{BMO(D)} \subset \mathrm{BMO_{loc}(D)} \subset \mathrm{FMO(D)}$ and as well-known $\mathrm{BMO_{loc}} \subset \mathrm{L^p_{loc}}$ for all $p \in [1, \infty)$. However, FMO is not a subclass of $\mathrm{L^p_{loc}}$ for any $p > 1$ but only of $\mathrm{L^1_{loc}}$. Thus, the class FMO is much more wider than $\mathrm{BMO_{loc}}$.

Versions of the next lemma has been first proved by us for the class BMO.

The main criteria in simply connected domains

Lemma 2.

Let D be a domain in $\mathbb{C}$ and let $\varphi : D \to \mathbb{R}$ be a non-negative function of the class $\mathrm{FMO}(z_0)$ for some $z_0 \in D$. Then

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{\varphi(z)\, dm(z)}{\left(|z-z_0| \log \frac{1}{|z-z_0|}\right)^2} = O\left(\log\log\frac{1}{\varepsilon}\right) \quad \text{as} \quad \varepsilon \to 0$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup\limits_{z \in D} |z - z_0|$.

Recall that we assume further that the dilatation quotients $K^T_\mu(z, z_0)$ and $K_\mu(z)$ are extended by 1 outside the domain D.

Choosing $\psi(t) = 1/\left(t \log\left(1/t\right)\right)$ in Lemma 1, see also Remark 1, we obtain by Lemma 2 the following result with the FMO type criterion.

The main criteria in simply connected domains

Theorem 1.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p(D)}$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L_1(D)}$, $\mathrm{K}_\mu^{\mathrm{T}}(\mathrm{z}, \mathrm{z}_0) \leqslant \mathrm{Q}_{\mathrm{z}_0}(\mathrm{z})$ a.e. in $\mathrm{U}_{\mathrm{z}_0}$ for each point $\mathrm{z}_0 \in \partial\mathrm{D}$, a neighborhood $\mathrm{U}_{\mathrm{z}_0}$ of z_0, a function $\mathrm{Q}_{\mathrm{z}_0} : \mathrm{U}_{\mathrm{z}_0} \to [0, \infty]$ in the class $\mathrm{FMO}(\mathrm{z}_0)$.

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $\mathrm{W}^{1,2}_{\mathrm{loc}}$ of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial\mathrm{D} \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = \mathrm{h} \circ \mathrm{f}$, $\mathrm{h} := \mathcal{A} + \mathrm{H}$, where $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D, $\mathrm{H} : \mathrm{D}_* \to \mathbb{C}$ is a generalized analytic function in $\mathrm{D}_* := \mathrm{f(D)}$ with the source S calculated in (9) and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition (10).

The main criteria in simply connected domains

By Corollary 1 we obtain the following consequence of Theorem 1.

Corollary 3.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p(D)}$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L_1(D)}$ and, for each point $\mathrm{z_0} \in \partial \mathrm{D}$,

$$\overline{\lim_{\varepsilon \to 0}} \int_{\mathrm{B(z_0,\varepsilon)}} \mathrm{K}^{\mathrm{T}}_\mu(\mathrm{z}, \mathrm{z_0}) \, \mathrm{dm(z)} \ < \ \infty \, .$$

Then all the conclusions of Theorem 1 on solutions for the Dirichlet problem (1) with arbitrary continuous boundary data $\varphi : \partial \mathrm{D} \to \mathbb{R}$ to the Beltrami equation (2) with the source σ hold.

The main criteria in simply connected domains

Since $K_\mu^T(z, z_0) \leqslant K_\mu(z)$ for all z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 1 with the BMO type criterion.

Corollary 4.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $Q \in \mathrm{BMO}$ in a neighborhood of ∂D. Then the conclusions of Theorem 1 hold.

Remark 5.

In particular, the conclusions of Theorem 1 hold if $Q \in W^{1,2}$ in a neighborhood of ∂D, because of $W^{1,2} \subset \mathrm{VMO}$.

The main criteria in simply connected domains

Corollary 5.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p(D)}$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $\mathrm{Q} \in \mathrm{FMO}$ in a neighborhood of $\partial\mathrm{D}$. Then the conclusions of Theorem 1 hold.

Similarly, choosing in Lemma 1 the function $\psi(\mathrm{t}) = 1/\mathrm{t}$, see also Remark 1, we come to the next statement with the Calderon-Zygmund type criterion.

Theorem 2.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p(D)}$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L_1(D)}$ and, for each point $\mathrm{z}_0 \in \partial\mathrm{D}$ and $\varepsilon_0 = \varepsilon(\mathrm{z}_0) > 0$,

The main criteria in simply connected domains

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon\to 0\,. \tag{15}$$

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{loc}$ of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = h\circ f$, $h := \mathcal{A} + H$, where $f : \mathbb{C}\to\mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D, $H : D_* \to \mathbb{C}$ is a generalized analytic function in $D_* := f(D)$ with the source S calculated in (9) and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition (10).

The main criteria in simply connected domains

Remark 6.

Choosing in Lemma 1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (15) by the conditions

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\mu^T(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall\, z_0 \in \partial D$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(z_0) > 0$. More generally, we would be able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form
$1/(t \log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Choosing in Lemma 1 the functional parameter $\psi_{z_0}(t) := 1/[t k_\mu^T(z_0,t)]$, where $k_\mu^T(z_0,r)$ is the integral mean of $K_\mu^T(z,z_0)$ over the circle $S(z_0,r) := \{z \in \mathbb{C} : |z-z_0| = r\}$, we obtain the Lehto type criterion.

The main criteria in simply connected domains

Theorem 3.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk_\mu^T(z_0, r)} = \infty \,.$$

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{loc}$ of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

The main criteria in simply connected domains

Moreover, $\omega = \mathrm{h} \circ \mathrm{f}$, $\mathrm{h} := \mathcal{A} + \mathrm{H}$, where $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D, $\mathrm{H} : \mathrm{D}_* \to \mathbb{C}$ is a generalized analytic function in $\mathrm{D}_* := \mathrm{f}(\mathrm{D})$ with the source S calculated in (9) and $\mathcal{A}$ is a holomorphic function in D_* with (10).

Corollary 6.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p}(\mathrm{D})$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L}_1(\mathrm{D})$ and, for each point $\mathrm{z}_0 \in \partial\mathrm{D}$,

$$\mathrm{k}_\mu^{\mathrm{T}}(\mathrm{z}_0, \varepsilon) = \mathrm{O}\left(\log \frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0\,. \tag{16}$$

Then all conclusions of Theorem 3 on solutions for the Dirichlet problem (1) with arbitrary continuous boundary data $\varphi : \partial\mathrm{D} \to \mathbb{R}$ to the Beltrami equation (2) with the source σ hold.

Remark 7.

In particular, the conclusions of Theorem 3 hold if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\, z_0 \in \partial D\,.$$

Moreover, the condition (16) can be replaced by the series of weaker conditions

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon} \cdot \ldots \cdot \log \ldots \log \frac{1}{\varepsilon}\right]\right) \qquad \forall\, z_0 \in \partial D\,.$$

Applying interconnections between integral conditions stated in V. Ryazanov, U. Srebro, E. Yakubov. Integral conditions in the theory of the Beltrami equations. Complex Var. Elliptic Equ. 57:12, 2012, 1247-1270, we obtain from Theorem 3 the following significant result with the Orlicz type criterion.

The main criteria in simply connected domains

Theorem 4.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$ and a neighborhood U_{z_0} of z_0,

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K_\mu^T(z, z_0)\right)\, dm(z) < \infty\ ,$$

where $\Phi_{z_0} : (0, \infty] \to (0, \infty]$ is a convex non-decreasing function such that

$$\int\limits_{\Delta(z_0)}^{\infty} \log\, \Phi_{z_0}(t)\, \frac{dt}{t^2} \ = \ +\infty \quad \text{for some } \Delta(z_0) > 0\ .$$

The main criteria in simply connected domains

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $\mathrm{W}^{1,2}_{\mathrm{loc}}$ of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial\mathrm{D} \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = \mathrm{h} \circ \mathrm{f}$, $\mathrm{h} := \mathcal{A} + \mathrm{H}$, where $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D, $\mathrm{H} : \mathrm{D}_* \to \mathbb{C}$ is a generalized analytic function in $\mathrm{D}_* := \mathrm{f}(\mathrm{D})$ with the source S calculated in (9) and $\mathcal{A}$ is a holomorphic function in D_* with the Dirichlet condition (10).

The main criteria in simply connected domains

Corollary 7.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in L_p(D)$, $p > 2$, with compact support, $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and, for each point $z_0 \in \partial D$, a neighborhood U_{z_0} of z_0 and $\alpha(z_0) > 0$,

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K_\mu^T(z,z_0)}\, dm(z) < \infty \, .$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1) with continuous data $\varphi : \partial D \to \mathbb{R}$ to the Beltrami equation (2) with the source σ hold.

The main criteria in simply connected domains

Corollary 8.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p(D)}$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of $\partial\mathrm{D}$,

$$\int\limits_{\mathrm{U}} \Phi\left(\mathrm{K}_\mu(\mathrm{z})\right) \mathrm{dm(z)} < \infty\,, \tag{17}$$

$\Phi : (0, \infty] \to (0, \infty]$ is a convex non-decreasing function for $\delta > 0$,

$$\int\limits_{\delta}^{\infty} \log\, \Phi(\mathrm{t})\, \frac{\mathrm{dt}}{\mathrm{t}^2} \;=\; +\infty\,. \tag{18}$$

All conclusions of Theorem 4 on solutions for the problem (1) with $\varphi : \partial\mathrm{D} \to \mathbb{R}$ to the Beltrami equation (2) with the source σ hold.

Remark 8.

By Theorems 2.5 and 5.1 in article mentioned before Theorem 4, condition (18) is not only sufficient but also necessary to have the regular solutions of the Dirichlet problem (1) in $\mathbb{D}$ for arbitrary Beltrami equations with sources (2), satisfying the integral constraints (17), for all continuous functions $\varphi : \partial\mathbb{D} \to \mathbb{R}$ because such solutions have the representation through regular homeomorphic solutions $\mathrm{f} = \mathrm{f}^{\mu}$ of the homogeneous Beltrami equation (5) from Proposition 1.

The main criteria in simply connected domains

Corollary 9.

Let D be a bounded simply connected domain in $\mathbb{C}$, $\sigma \in \mathrm{L_p}(\mathrm{D})$, $\mathrm{p} > 2$, with compact support, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of $\partial\mathrm{D}$ and $\alpha > 0$,

$$\int\limits_{\mathrm{U}} \mathrm{e}^{\alpha \mathrm{K}_\mu(\mathrm{z})}\, \mathrm{dm}(\mathrm{z}) \ < \ \infty \, .$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1) with continuous data $\varphi : \partial\mathrm{D} \to \mathbb{R}$ to the Beltrami equation (2) with the source σ hold.

On multi-valued solutions in general domains

In this section we formulate the main lemma and its consequences on the existence, representation and regularity of the so-called multi-valued solutions ω of the Dirichlet problem (1) to the Beltrami equations with sources (2) in the spirit of the theory of multi-valued analytic functions in arbitrary bounded domains D in $\mathbb{C}$ with no boundary component degenerated to a single point.

We say that a locally Hölder continuous function $\omega : \mathrm{B}(z_0, \varepsilon_0) \to \mathbb{C}$, where $\mathrm{B}(z_0, \varepsilon_0) \subseteq \mathrm{D}$, is a **local regular solution of the equation** (2) if $\omega \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ and ω satisfies (2) a.e. in $\mathrm{B}(z_0, \varepsilon_0)$. Local regular solutions $\omega_0 : \mathrm{B}(z_0, \varepsilon_0) \to \mathbb{C}$ and $\omega_* : \mathrm{B}(z_*, \varepsilon_*) \to \mathbb{C}$ of the equation (2) will be called extension of each to other if there is a finite chain of its local regular solutions $\omega_i : \mathrm{B}(z_i, \varepsilon_i) \to \mathbb{C}$, $i = 1, \ldots, m$, such that $\omega_1 = \omega_0$, $\omega_m = \omega_*$ and $\omega_i(z) = \omega_{i+1}(z)$ for all points $z \in E_i := \mathrm{B}(z_i, \varepsilon_i) \cap \mathrm{B}(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset$, $i = 1, \ldots, m-1$.

On multi-valued solutions in general domains

A collection of local regular solutions $\omega_j : B(z_j, \varepsilon_j) \to \mathbb{C}$, $j \in J$, will be called a **regular multi-valued solution** of the equation (2) in D if the collection of the disks $B(z_j, \varepsilon_j)$ cover the domain D and ω_j are extensions of each to other through this collection and the collection is maximal by inclusion.

A regular multi-valued solution of the equation (2) will be called a **regular multi-valued solution of the Dirichlet problem** (1) to (2) in D if $u(z) = \mathrm{Re}\,\omega(z) = \mathrm{Re}\,\omega_j(z)$, $z \in B(z_j, \varepsilon_j)$, $j \in J$, is a single-valued function in D satisfying the Dirichlet condition $\lim\limits_{z \in \zeta} u(z) = \varphi(\zeta)$ for all $\zeta \in \partial D$.

On multi-valued solutions in general domains

Lemma 3.

Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\sigma \in \mathrm{L_p}(\mathrm{D})$, $\mathrm{p} > 2$, with compact support in D, $\mu : \mathrm{D} \to \mathbb{C}$ be a measurable function with $|\mu(\mathrm{z})| < 1$ a.e., K_μ be locally bounded in D, $\mathrm{K}_\mu \in \mathrm{L}_1(\mathrm{D})$ and, for each point $\mathrm{z}_0 \in \partial\mathrm{D}$ and $\varepsilon_0 = \varepsilon(\mathrm{z}_0) > 0$,

$$\int\limits_{\varepsilon<|\mathrm{z}-\mathrm{z}_0|<\varepsilon_0} \mathrm{K}_\mu^{\mathrm{T}}(\mathrm{z},\mathrm{z}_0) \cdot \psi^2_{\mathrm{z}_0,\varepsilon}(|\mathrm{z}-\mathrm{z}_0|)\,\mathrm{dm}(\mathrm{z}) = \mathrm{o}(\mathrm{I}^2_{\mathrm{z}_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0\,,$$

where $\psi_{\mathrm{z}_0,\varepsilon} : (0,\varepsilon_0) \to (0,\infty)$ is a family of measurable functions such that

$$\mathrm{I}_{\mathrm{z}_0}(\varepsilon) \colon = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{\mathrm{z}_0,\varepsilon}(\mathrm{t})\,\mathrm{dt} < \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\,.$$

On multi-valued solutions in general domains

Then the Beltrami equation (2) with the source σ has a regular multi-valued solution ω of the Dirichlet problem (1) in D for each continuous function $\varphi : \partial\mathrm{D} \to \mathbb{R}$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = \mathrm{h} \circ \mathrm{f}$, $\mathrm{h} := \mathcal{A} + \mathrm{H}$, where $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ is a $\mu-$conformal mapping with μ extended by zero outside D, $\mathrm{H} : \mathrm{D}_* \to \mathbb{C}$ is a generalized analytic function in $\mathrm{D}_* := \mathrm{f}(\mathrm{D})$ with the source S calculated in (9) and $\mathcal{A}$ is a multi-valued analytic function in D_* with a single valued real part satisfying the Dirichlet condition (10).

Remark 9.

In this case, Lemma 3 implies the same integral criteria as in Theorems 1-4, Corollaries 3-9, Remarks 5-8 in terms of dilatations K_μ and $\mathrm{K}^{\mathrm{T}}_\mu$. Hence we do not give them in the explicit form and further comments here.

On multi-valued solutions in general domains

Let us denote by $\mathbb{S}^{2\times 2}$ the collection of all 2×2 matrices with real entries

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

which are symmetric, i.e., $a_{12} = a_{21}$, with $\det A = 1$ and **ellipticity condition** $\det (I + A) > 0$, where I is the unit 2×2 matrix. The latter condition means in terms of entries of A that $(1 + a_{11})(1 + a_{22}) > a_{12}a_{21}$.

Now, let us consider in a domain D of the complex plane $\mathbb{C}$ the Poisson type equations (4), where $A : D \to \mathbb{S}^{2\times 2}$ is a measurable matrix valued function whose elements $a_{ij}(z)$, $i, j = 1, 2$ are measurable and locally bounded and first the source $g : D \to \mathbb{R}$ is a scalar function in $L_{1,loc}$.

It is well–known, see Theorem 16.1.6 in the monograph K. Astala, T. Iwaniec, G.J. Martin (2009) Elliptic differential equations and quasiconformal mappings in the plane. Princeton Math. Ser. 48. Princeton: Princeton Univ. Press, nonhomogeneous Beltrami equations (2) with locally bounded K_μ are closely connected with the Poisson type equations (4), where $\mathrm{A} : \mathrm{D} \to \mathbb{S}^{2\times 2}$ is the measurable matrix valued function

$$\mathrm{A}(\mathrm{z}) \ := \left[\begin{array}{cc} \frac{|1-\mu(\mathrm{z})|^2}{1-|\mu(\mathrm{z})|^2} & \frac{-2\mathrm{Im}\,\mu(\mathrm{z})}{1-|\mu(\mathrm{z})|^2} \\ \frac{-2\mathrm{Im}\,\mu(\mathrm{z})}{1-|\mu(\mathrm{z})|^2} & \frac{|1+\mu(\mathrm{z})|^2}{1-|\mu(\mathrm{z})|^2} \end{array} \right] ,$$

whose entries $\mathrm{a_{ij}}(\mathrm{z})$ are dominated by $\mathrm{K}_\mu(\mathrm{z})$ and, thus, they are locally bounded.

Vice versa, locally uniform elliptic (4) with measurable $\mathrm{A} : \mathrm{D} \to \mathbb{S}^{2\times 2}$ just correspond to nonhomogeneous Beltrami equations (2) with coefficients

$$\mu_{\mathrm{A}} \ := \ -\frac{\mathrm{a}_{11} - \mathrm{a}_{22} + \mathrm{i}(\mathrm{a}_{12} + \mathrm{a}_{21})}{2 + \mathrm{a}_{11} + \mathrm{a}_{22}} \ ,$$

whose dilatation quotients $\mathrm{K}_{\mu_{\mathrm{A}}}$ are locally bounded.

As it was before, we assume here that the dilatations $\mathrm{K}^{\mathrm{T}}_{\mu_{\mathrm{A}}}(\mathrm{z}, \mathrm{z}_0)$ and $\mathrm{K}_{\mu_{\mathrm{A}}}(\mathrm{z})$ are extended by 1 outside of the domain D.

Lemma 4.

Let D be a bounded domain in $\mathbb{C}$ with no boundary component degenerated to a single point, $\mathrm{g} \in \mathrm{L}_{\mathrm{p}'}(\mathrm{D})$, $\mathrm{p}' > 1$, with compact support, and $\mathrm{A} : \mathrm{D} \to \mathbb{S}^{2\times 2}$ be a matrix function with its locally Hölder continuous entries.

Suppose also that $\mathrm{K}_{\mu_\mathrm{A}} \in \mathrm{L}^1(\mathrm{D})$ and, for each $\mathrm{z}_0 \in \partial\mathrm{D}$,

$$\int\limits_{\varepsilon<|\mathrm{z}-\mathrm{z}_0|<\varepsilon_0} \mathrm{K}^{\mathrm{T}}_{\mu_\mathrm{A}}(\mathrm{z},\mathrm{z}_0)\cdot\psi^2_{\mathrm{z}_0,\varepsilon}(|\mathrm{z}-\mathrm{z}_0|)\,\mathrm{dm}(\mathrm{z}) = \mathrm{o}(\mathrm{I}^2_{\mathrm{z}_0}(\varepsilon)) \qquad \text{as } \varepsilon\to 0\,,$$

where $\varepsilon_0 = \varepsilon(\mathrm{z}_0) > 0$ and $\psi_{\mathrm{z}_0,\varepsilon} : (0,\varepsilon_0) \to (0,\infty)$ are measurable functions with

$$\mathrm{I}_{\mathrm{z}_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{\mathrm{z}_0,\varepsilon}(\mathrm{t})\,\mathrm{dt} < \infty \qquad \forall\, \varepsilon\in(0,\varepsilon_0)\,.$$

Dirichlet problem for Poisson type equations

Then the Poisson type equation (4) has the unique weak continuous solution u in the class $C^{\alpha}_{loc} \cap W^{1,p}_{loc} \cap W^{2,p'}_{loc}$, $\alpha = 1 - 2/p$, for some $p > 2$ of the Dirichlet problem (3) in D for each continuous function $\varphi : \partial D \to \mathbb{R}$.

Moreover, $u = U \circ f$, where $f : \mathbb{C} \to \mathbb{C}$ is a μ_A-conformal mapping with μ_A extended by zero outside D, U is a weak generalized harmonic function with the source G of the class $L_{p'}(D_*)$ in the domain $D_* := f(D)$,

$$G \ := \ \frac{g}{J} \circ f^{-1} \ , \quad J(z) := |f_z|^2 - |f_{\overline{z}}|^2 \ ,$$

of the class $C^{\alpha}_{loc} \cap W^{1,p}_{loc} \cap W^{2,p'}_{loc}$ in D_*, satisfying the Dirichlet condition

$$\lim_{w \to \zeta} \ U(w) = \varphi_*(\zeta) \quad \forall\, \zeta \in \partial D_* \ , \quad \varphi_* := \varphi \circ f^{-1}|_{\partial D_*} \ .$$

Dirichlet problem for Poisson type equations

Here u is called a **weak solution** of the Poisson type equation (4) if

$$\int\limits_{\mathrm{D}} \{\langle \mathrm{A}(\mathrm{z})\nabla \mathrm{u}(\mathrm{z}), \nabla\psi\rangle \ + \ \mathrm{g}(\mathrm{z})\,\psi(\mathrm{z})\}\,\mathrm{dm}(\mathrm{z}) \ = \ 0 \quad \forall\,\psi \in \mathrm{C}_0^1(\mathrm{D})\,.$$

Remark 10.

In turn, $\mathrm{U} := \mathcal{H} + \mathcal{N}^{\mathrm{G}}$, where $\mathcal{H}$ is the unique harmonic function in D_*, satisfying the Dirichlet condition

$$\lim_{\mathrm{w}\to\zeta} \mathcal{H}(\mathrm{w}) = \varphi^*(\zeta) \quad \forall\,\zeta \in \partial\mathrm{D}_*\,, \quad \varphi^* := \varphi_* - \mathcal{N}^{\mathrm{G}}|_{\partial\mathrm{D}_*}\,,$$

$$\mathcal{N}^{\mathrm{G}}(\mathrm{z}) \ := \ \frac{1}{2\pi}\int\limits_{\mathrm{D}_*} \mathrm{G}(\mathrm{w})\,\ln|\mathrm{z}-\mathrm{w}|\,\mathrm{d\,m}(\mathrm{w})\,.$$

Furthermore, if $\mathrm{g} \in \mathrm{L}_{\mathrm{p}'}(\mathrm{D})$ for $\mathrm{p}' > 2$, then $\mathrm{u} \in \mathrm{C}^{1,\alpha'}_{\mathrm{loc}}(\mathrm{D})$ with $\alpha' = 1 - 2/\mathrm{p}'$.

Lemma 4 implies the same integral criteria as in Theorems 1-4, Corollaries 3-9, Remarks 5-8 from Section 3 in terms of dilatations K_μ and $\mathrm{K}_\mu^{\mathrm{T}}$ with $\mu = \mu_{\mathrm{A}}$. Hence we do not give them in the explicit form and further comments here.

As a result, we have a number of effective integral criteria for the solvability of the classical Dirichlet problem (3) in the most general admissible domains to one of the main equations (4) of the hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media.

A short description of the rest of results

In [1], it was established necessary as well as sufficient conditions for the cavitation at isolated singular points of the Beltrami equations (5) and existence of continuous solutions of the relevant $\mathrm{A}-$harmonic equations $\mathrm{div}[\mathrm{A}(\mathrm{z})\nabla\,\mathrm{u}(\mathrm{z})] = 0$ in Section 5 independently on the cavitation at the isolated singular points of equations (5).

The papers [4] and [5] contain results on existence, representation and regularity of homeomorphic solutions with hydrodynamic normalization at infinity and of solutions of the Dirichlet problem, correspondingly, to the degenerate Beltrami equations (5).

In [2], it was proved existence, representation and regularity of solutions to the semi-linear nondegenerate Beltrami equations with nonlinear sources and the relevant semi-linear Poisson type equations describing various physical phenomena in anisotropic and inhomogeneous media.

A short description of the rest of results

The article [6] extends results on existence, representation and regularity of solutions for the Dirichlet problem to the semi-linear Beltrami equations degenerated on the boundary with nonlinear sources and to the relevant degenerate semi-linear Poisson type equations in strongly anisotropic and inhomogeneous media.

Finally, the article [7] includes the corresponding results on existence, representation and regularity of nonclassical solutions of the Hilbert and Riemann problems with boundary dates that are measurable with respect to logarithmic capacity for the Beltrami equations with sources.

It was published with references to the fund ALLEA:

1. Gutlyanskii, V., Martio, O. and Ryazanov, V. A-harmonic equation and cavitation. Ann. Fenn. Math.48 (2023), no.1, 277-297. **(Q1)** *https://doi.org/10.54330/afm.127639*

2. Gutlyanskii, V., Ryazanov, V., Nesmelova O., Yakubov, E. Toward the theory of semi-linear Beltrami equations. Constructive Mathematical Analysis 6 (2023), no. 3, 151-163. **(Q2)** *DOI: 10.33205/cma.1248692*

3. Gutlyanskii, V., Ryazanov, V., Nesmelova O., Yakubov, E. The Dirichlet problem for the Beltrami equations with sources. J. Math. Sci., 273 (2023), no. 3, 351-376. **(Q3)** *DOI: 10.1007/s10958-023-06503-0*

4. Gutlyanskii, V. Ya., Ryazanov, V. I., Sevost'yanov, E. A., Yakubov, E. Hydrodynamic normalization conditions in the theory of degenerate Beltrami equations. Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki (2023), no. 2, 10-17. *https://doi.org/10.15407/dopovidi2023.02.010*

Publications with references to the fund ALLEA:

5. Gutlyanskii, V.Ya., Ryazanov, V.I., Sevost'yanov, E.A., Yakubov, E. On the Dirichlet problem for degenerate Beltrami equations. Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki (2023), no. 3, 9-16.
https://doi.org/10.15407/dopovidi2023.03.009

It is accepted for publication with references to the fund ALLEA:

6. Gutlyanskii, V., Ryazanov, V., Nesmelova O., Yakubov, E. BMO and Dirichlet problem for semi-linear equations in the plane. Topological Methods in Nonlinear Analysis, 38pp. **(Q2)**

It was submitted for publication with references to fund ALLEA:

7. Gutlyanskii, V., Ryazanov, V., Nesmelova O., Yakubov, E. On the Hilbert and Riemann problems for Beltrami equations with sources. Annals of PDE, 19 pp. **(Q1)**

Thank you for your attention!