Semimetric Spaces and Combinatorial Similarities

(Supported by the Grant EFDS-FL2-08 of the found The European Federation of Academies of Sciences and Humanities (ALLEA))

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IAMM of NASU

November 2023

A *metric* on a set X is a function $d\colon X^2 \to \mathbb{R}$ such that for all x, y, $z \in X$:

1. $d(x,y) \geqslant 0$ with equality if and only if $x = y$, the *positivity property*;
2. $d(x,y) = d(y,x)$, the *symmetry property*;
3. $d(x,y) \leq d(x,z) + d(z,y)$, the *triangle inequality*.

Here and what follows X^2 is the Cartesian square of the set X, i.e., X^2 is the set of all ordered pairs $< x, y >$, where x, $y \in X$.

Definition

Let X be a set and let $d\colon X^2 \to \mathbb{R}$ be a symmetric function. The function d is a *semimetric* on X if it satisfies the positivity property.

If d is a semimetric on X, we say that (X, d) is a *semimetric space*.

In the next definition we will denote by $d(X^2)$ the range of d,

$$d(X^2) := \{d(x,y) \colon x, y \in X\}.$$

Definition

Let (X,d) and (Y,ρ) be semimetric spaces. The spaces (X,d) and (Y,ρ) are *combinatorially similar* if there exist bijections $\Psi \colon Y \to X$ and $f \colon d(X^2) \to \rho(Y^2)$ such that

$$\rho(x,y) = f\big(d(\Psi(x), \Psi(y))\big) \tag{1}$$

for all x, $y \in Y$. In this case, we will say that $\Psi \colon Y \to X$ is a *combinatorial similarity*.

For every semimetric space (X,d), the set of all combinatorial self similarities $X \to X$ is a subgroup of the symmetric group $\mathbf{Sym}(X)$ of all permutations on X. In what follows this subgroup will be denoted as $\mathbf{Cs}(X,d)$.

Example

Semimetric spaces (X, d) and (Y, ρ) are *isometric* if there is a bijection $\Psi\colon Y \to X$ such that $\rho(x, y) = d(\Psi(x), \Psi(y))$ for all x, $y \in Y$. In this case we will say that Ψ is an *isometry* of (X, d) and (Y, ρ). It is easy to see that all isometrics are combinatorial similarities. Moreover, a combinatorial similarity $\Psi\colon Y \to X$ is an isometry if and only if (1) holds for all x, $y \in Y$ with $f(t) = t$ for every $t \in d(X^2)$.

The group of all self isometries of a semimetric space (X, d) will be denoted as $\mathbf{Iso}(X, d)$.

Example

Let $\Phi\colon X \to Y$ be a bijection and let $d\colon X^2 \to \mathbb{R}$ and $\rho\colon Y^2 \to \mathbb{R}$ be some semimetrics. The mapping Φ is a *weak similarity* of (X, d) and (Y, ρ) if and only if the equivalence

$$(d(x, y) \leqslant d(w, z)) \Leftrightarrow (\rho(\Phi(x), \Phi(y)) \leqslant \rho(\Phi(w), \Phi(z))) \tag{2}$$

is valid for all x, y, z, $w \in X$.

Let (X, d) be a metric space. The metric d is said to be *strongly rigid* if, for all x, y, u, $v \in X$, the condition

$$d(x, y) = d(u, v) \neq 0 \tag{3}$$

implies

$$(x = u \text{ and } y = v) \text{ or } (x = v \text{ and } y = u). \tag{4}$$

Definition

Let (X, d) be a semimetric space. The semimetric d is *strongly rigid* if (3) implies (4) for all x, y, u, $v \in X$.

Definition

A semimetric space (X, d) is *weakly rigid* if every three-point subspace of (X, d) is strongly rigid.

We say that a semimetric $d\colon X^2 \to \mathbb{R}$ is *discrete* if the inequality

$$|d(X^2)| \leqslant 2$$

holds, where $|d(X^2)|$ is the cardinal number of the set $d(X^2)$. Thus, a semimetric $d\colon X^2 \to \mathbb{R}$ is discrete if and only if there is $k > 0$ such that the equality

$$d(x, y) = k$$

holds for all different x, $y \in X$. It follows directly from the definitions that a semimetric space (X, d) is discrete if and only if

$$\mathbf{Iso}(X, d) = \mathbf{Sym}(X)$$

holds.

The next Proposition gives a characterization of discrete $d\colon X^2 \to \mathbb{R}$ in terms of combinatorial self similarities of (X, d).

Proposition

Let (X, d) be a semimetric space with $|X| \geqslant 3$. Then the following statements are equivalent:

1. *(X, d) is discrete.*
2. *(X, d) contains an equilateral triangle and the equality $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ holds.*

Theorem

Let (X, d) be a nonempty semimetric space. Then the following statements are equivalent:

1. *At least one of the following conditions has been fulfilled:*
   (i) (X, d) is strongly rigid;
   (ii) (X, d) is discrete;
   (iii) (X, d) is weakly rigid and all three-point subspaces of (X, d) are isometric.
2. $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ *holds.*

Corollary

The following conditions are equivalent for every nonempty set X:

1. $|X| = 4$.
2. *There is a semimetric $d\colon X^2 \to \mathbb{R}$ such that $\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$ but d is neither strongly rigid nor discrete.*

Corollary

Let (X, d) be a semimetric space such that the inequality

$$|X| > \mathfrak{c}$$

holds, where $\mathfrak{c}$ is the cardinality of the continuum. Then

$$\mathbf{Cs}(X, d) = \mathbf{Sym}(X)$$

holds if and only if

$$\mathbf{Iso}(X, d) = \mathbf{Sym}(X).$$

Example

Let $Z = \{z_1, z_2, z_3, z_4\}$ be the four-point subset of the complex plane $\mathbb{C}$,

$$z_1 = 0 + 0i, \quad z_2 = 0 + 3i, \quad z_3 = 4 + 3i, \quad z_4 = 4 + 0i,$$

and let d be the restriction of the usual Euclidean metric on Z^2. The equalities $d(z_1, z_2) = d(z_3, z_4) = 3$ and $d(z_1, z_4) = d(z_2, z_3) = 4$ imply that (Z, d) is neither strongly rigid nor discrete but it can be proved directly that $\mathbf{Cs}(Z, d) = \mathbf{Sym}(Z)$.

The next example is an example of a four-point metric space (Y, ρ) satisfying the equality $\mathbf{Cs}(Y, \rho) = \mathbf{Sym}(Y)$.

Example

A four-point metric space (X, ρ) is called a *pseudolinear quadruple* if for a suitable enumeration of the points we have

$$\rho(x_1, x_2) = \rho(x_3, x_4) = s, \quad \rho(x_2, x_3) = \rho(x_4, x_1) = t, \quad \rho(x_2, x_4) = \rho(x_3, x_1) = s + t,$$

with some positive reals s and t. The pseudolinear quadruple (X, ρ) is combinatorially similar to the rectangle from previous example iff $s \neq t$.

List of Publications

- *V. Bilet*, *O. Dovgoshey*, Completeness, closedness and metric reflections of pseudometric spaces // Topology and its Applications, V. 327, 2023: 108 – 440.
- *V. Bilet*, *O. Dovgoshey*, When all permutations are combinatorial similarities // Bull. Corean. Math. Soc. 2023, 60(3): 733 – 746.
- *V. Bilet*, *O. Dovgoshey*, From minimality to maximality via metric reflection // International Science Conference "Algebraic and Geometric Methods of Analysis" devoted to the anniversary of Dmytro Grave (1863 - 1939), May 29 – June 1, 2023, Odesa, Ukraine. Book of Abstacts, P. 12 – 14.

THANK YOU FOR ATTENTION!