On the second correlation function of the characteristic polynomials of sparse Non-Hermitian matrices

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Random matrices

Definition

A sequence M_n of $n \times n$ matrices, which entries are random variables, is called a random matrix ensemble.

Examples

- Non-Hermitian random matrices with independent entries (Ginibre ensemble if entries are Gaussian)

$$M_{jk} = n^{-1/2} w_{jk},$$

$$\{w_{jk}\} \text{ — i.i.d.}, \quad \mathbf{E}\{w_{jk}\} = 0, \quad \mathbf{E}\{|w_{jk}|^2\} = 1, \quad \mathbf{E}\{w_{jk}^2\} = 0$$

- The adjacency matrices of random Erdős–Rényi graphs

$$M_{jk} = M_{kj} = \begin{cases} 1 - \delta_{jk} & \text{with probability } \frac{p_n}{n}; \\ 0 & \text{with probability } 1 - \frac{p_n}{n}. \end{cases}$$

Global regime

Normalized counting measure of eigenvalues (NCM) and linear eigenvalue statistics

$$N_n(\Delta) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\Delta}(\lambda_j^{(n)}), \quad \mathcal{N}_n[\varphi] = \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) = \operatorname{tr}\varphi(M)$$

Questions

1. $N_n(\Delta) \xrightarrow{w} N(\Delta)$
2. Central Limit Theorem for linear eigenvalue statistics

$\sigma = \operatorname{supp} N$ is called a spectrum.

For the Ginibre ensemble

$$N(\Delta) = \int_{\Delta} \rho_c(z) d\bar{z}\, dz, \quad \rho_c(z) = \frac{1}{\pi} \cdot \mathbb{1}_{|z|<1}.$$

Local regime

The spectral correlation functions $R_m^{(n)}(z_1, \dots, z_m)$ are defined by

$$\mathbf{E}\left\{ \sum_{1 \le j_1 \ne \dots \ne j_m \le n} \eta\left(\lambda_{j_1}^{(n)}, \dots, \lambda_{j_m}^{(n)}\right) \right\}$$
$$= \int_{\mathbb{C}^m} \eta(z_1, \dots, z_m) R_m^{(n)}(z_1, \dots, z_m) d\bar{z}_1 dz_1 \cdots d\bar{z}_m dz_m,$$

where η is any bounded continuous symmetric function.

Universality conjecture

$$\lim_{n \to \infty} \pi R_m^{(n)}\left(z_0 + \frac{\zeta_1}{n}, \dots, z_0 + \frac{\zeta_m}{n}\right) = \det\left\{ e^{-\frac{|\zeta_j|}{2} - \frac{|\zeta_k|}{2} + \zeta_j \overline{\zeta_k}} \right\}_{j,k=1}^{m}$$

The correlation functions of the characteristic polynomials

Let M_n be some ensemble of non-Hermitian random matrices. Consider the m^{th} correlation function of characteristic polynomials

$$\mathsf{f}_m(Z) = \mathbf{E}\left\{ \prod_{j=1}^{m} \det\left(M_n - z_j\right)\left(M_n - z_j\right)^* \right\}, \quad Z = \operatorname{diag}\{z_j\}_{j=1}^m$$

$$z_j = z_0 + \frac{\zeta_j}{\sqrt{n}}, \qquad |z_0| < 1$$

$$\mathsf{f}_m(Z) \xrightarrow[n\to\infty]{} ?$$

Some results

- Brezin and Hikami (2000, 2001)
- Strahov and Fyodorov (2002, 2003); Fyodorov and Khoruzhenko (2006)
- T. Shcherbina (2011, 2013, 2014, 2015, 2016, 2017)
- Recher, Kieburg, Guhr, and Zirnbauer (2012)
- Akemann and Vernizzi (2003); Webb and Wong (2019)
- A. (2019, 2020, 2022)

Sparse random matrices

Ensemble of sparse non-Hermitian random matrices

$$M_n = (d_{jk} w_{jk})_{j,k=1}^{n},$$

where

$$d_{jk} = p^{-1/2} \begin{cases} 1 \text{ with probability } \frac{p}{n}; \\ 0 \text{ with probability } 1 - \frac{p}{n}. \end{cases}$$

and w_{jk} are independent complex Gaussian random variables with zero mean such that

$$\mathbf{E}\{|w_{jk}|^2\} = 1.$$

The results on the ensemble of sparse random matrices

- For $p \to \infty$ the normalized counting measure is the same as for the Ginibre ensemble.
  - Götze, Tikhomirov (2010) for $p > n^{3/4+\varepsilon}$;
  - Tau, Vu (2008) for $p > n^{\varepsilon}$;
  - Basak, Rudelson (2019) for $p > (\log n)^2$;
  - Rudelson, Tikhomirov (2019) for any $p \to \infty$.
- For finite p the convergence of the normalized counting measure was proven by Sah, Sahasrabudhe, Sawhney (2023).

The second order correlation function

Theorem (A., Shcherbina: 23)

Consider the normalized second order correlation function

$$D_2(Z) = \frac{\mathsf{f}_2(Z)}{\sqrt{\mathsf{f}_2(z_1 I)\mathsf{f}_2(z_2 I)}}.$$

Then we have for finite p

(i) for $p > 2$

$$\lim_{n\to\infty} D_2\left(Z\right) = \frac{1 - e^{-\alpha|\zeta_1 - \zeta_2|}}{\alpha\left|\zeta_1 - \zeta_2\right|},$$

where $\alpha \in [0, 1]$ is a solution to the equation

$$p\alpha - p + 2 = p\left|z_0\right|^2 (1-\alpha)^2 (p\left|z_0\right|^2 (1-\alpha) + 2).$$

.

The second order correlation function

(ii) for $1 < p \leq 2$

$$\lim_{n\to\infty} D_2\left(Z\right) = \begin{cases} \frac{1-e^{-\alpha|\zeta_1-\zeta_2|}}{\alpha|\zeta_1-\zeta_2|}, & \text{if } r_* < \left|z_0\right|^2 < 1, \\ 1, & \text{if } \left|z_0\right|^2 < r_*; \end{cases}$$

where $r_* = \frac{1-\sqrt{p-1}}{p}$;

(ii) for $p \leq 1$

$$\lim_{n\to\infty} D_2\left(Z\right) = 1.$$

Grassmann variables

Let $\{\psi_j, \overline{\psi}_j\}_{j=1}^n$ be a set of anti-commuting variables, i.e.

$$\psi_j\psi_k + \psi_k\psi_j = \overline{\psi}_j\psi_k + \psi_k\overline{\psi}_j = \overline{\psi}_j\overline{\psi}_k + \overline{\psi}_k\overline{\psi}_j = 0.$$

Then for an arbitrary $n \times n$ matrix A

$$\int \exp\left\{ -\sum_{j,k=1}^{n} \overline{\psi}_j A_{jk}\psi_k \right\} \prod_{j=1}^{n} d\overline{\psi}_j d\psi_j = \det A.$$

List of publications

- I. Afanasiev, T. Shcherbina, On the second correlation function of the characteristic polynomials of sparse Non-Hermitian matrices, to submit to Journal of Mathematical Physics, Analysis, Geometry.

Thank you for attention!