Interface propagation properties for a nonlocal thin-film equation

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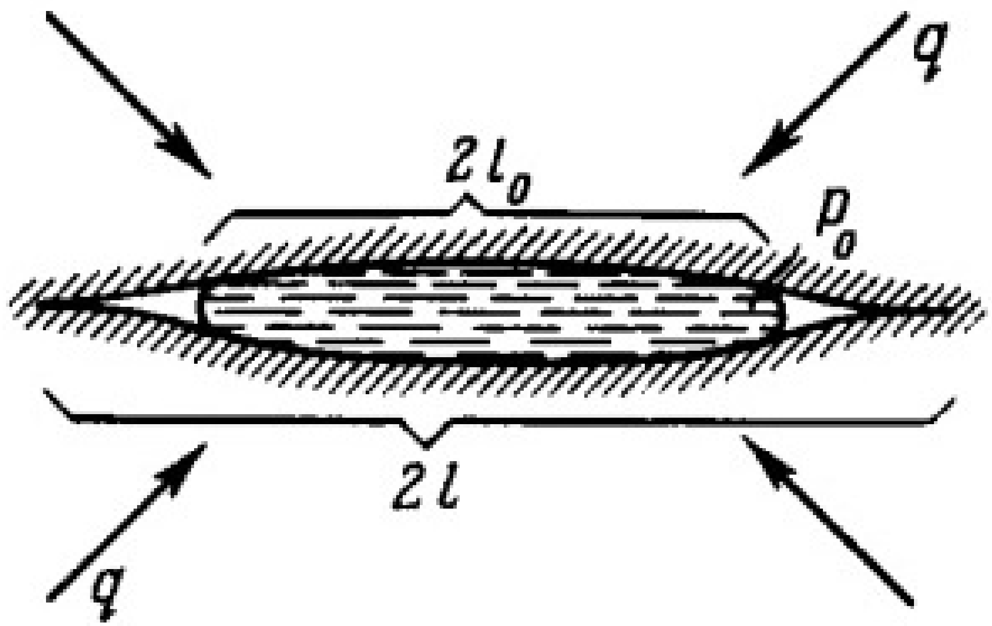
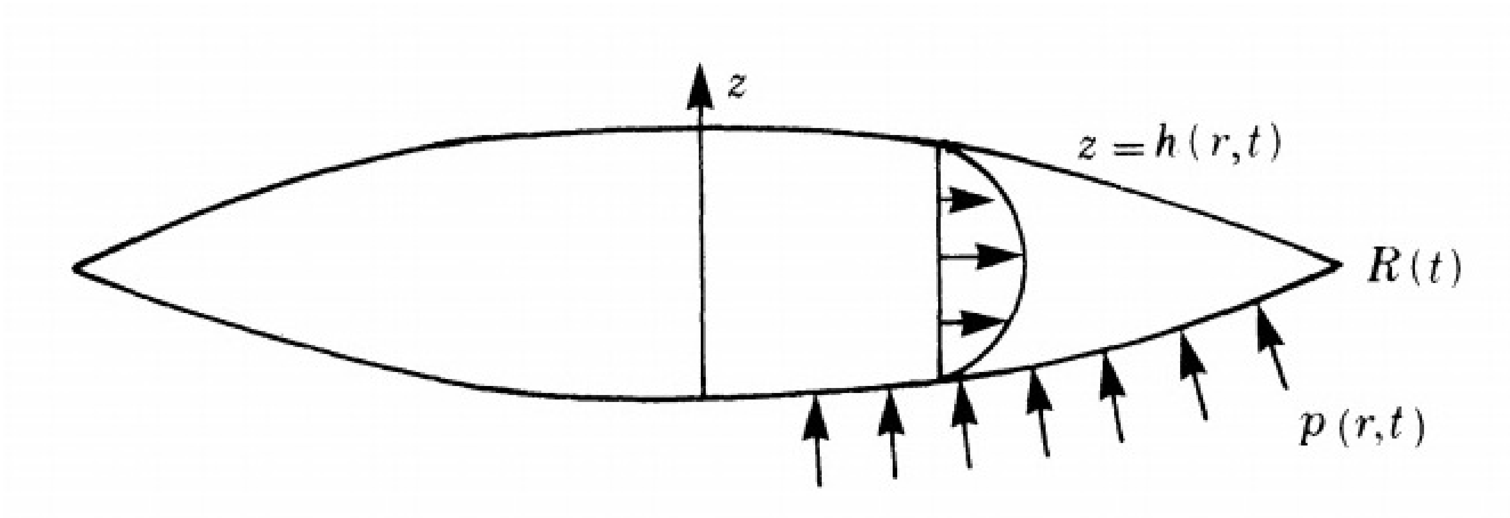
Cracks in rocks



The investigation of crack extension in rock massifs is of great interest in theoretical geology. Cracks can form in rocks because of various causes of tectonic character, but also because of some artificial actions (mining excavations, hydraulic fracture of oil-bearing strata, etc.).

Mathematical model



lubrication idea: $\frac{h}{R} \ll 1$

pressure: $p(r,t) = -\frac{E}{4\pi}\int \frac{h_z(z,t)\,dz}{r-z} = \frac{E}{4}(-\Delta)^{\frac{1}{2}}h(r,t)$

- **S. Khristianovich, Y.P. Zheltov. Formation of vertical fractures by means of highly viscous liquid, In World Petroleum Congress Proceedings, 579-586 (1955)**

- **J. Geertsma, F. De Klerk, et al. A rapid method of predicting width and extent of hydraulically induced fractures. Journal of petroleum technology, 21(12):1-571 (1969)**

Mathematical model

$$h_t = (|h|^n (p(h))_x)_x, \; p(h) = (-\Delta)^s h,$$

where $n > 0$ **and** $(-\Delta)^s$ **denotes the fractional Laplacian of order** $s \in (0, 1)$**.**

$s = 0$ – **porous medium equation,** $s = 1$ – **thin film equation**

- **D. A. Spence and P. Sharp. Self-similar solutions for elastohydrodynamic cavity flow. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 400(1819):289-313 (1985)** ($n = 3,\ s = \frac{1}{2}$)
- **C. Imbert and A. Mellet. Existence of solutions for a higher order non-local equation appearing in crack dynamics. Nonlinearity, 24(12):3487-3514 (2011)** ($n \geq 1,\ s = \frac{1}{2}$)
- **R. Tarhini. Study of a family of higher order nonlocal degenerate parabolic equations: from the porous medium equation to the thin film equation. J. Differential Equations, 259(11):5782-5812 (2015)** ($n \geq 1,\ s \in (0, 1)$)

Weak generalised solutions

[Imbert and Mellet, 2011], [Tarhini, 2015]

$$h_t = (|h|^n (p(h))_x)_x \textbf{ and } p(h) = (-\Delta)^s h \textbf{ in } \Omega_T,$$
$$h_x = |h|^n (p(h))_x = 0 \textbf{ on } \partial\Omega_T,$$
$$h(x,0) = h_0(x) \geq 0, \qquad h_0 \in H^s(\Omega).$$

- **weak generalised solution**

$$\iint_{\Omega_T} h\phi_t - n \iint_{\Omega_T} h^{n-1} h_x p(h) \phi_x$$
$$- \iint_{\Omega_T} h^n p(h) \phi_{xx} + \int_\Omega h_0 \phi(x,0) = 0$$

for all $\phi \in \mathcal{D}(\bar{\Omega} \times [0,T))$ **such that** $\phi_x = 0$ **on** $\partial\Omega \times (0,T)$**,**

$$0 \leq h \in L^\infty(0,T; H^s(\Omega)) \cap L^2(0,T; H_N^{s+1}(\Omega)).$$

Finite speed of propagation

Theorem 1 .
Let us assume $\Omega = (0,1)$***,*** $s \in (0,1)$***,*** $n \in (1,2)$***, and***

$$\mathrm{supp}(h_0(\cdot)) \subseteq [1/2 - r_0, 1/2 + r_0]$$

for some $r_0 \in (0, 1/2)$***. Let*** h ***be a weak generalised solution. Then there exists a time*** $T^* > 0$ ***and a non-decreasing function*** $d(t) \in C([0,T^*])$***, with*** $d(0) = r_0$***, such that***

$$\mathrm{supp}(h(\cdot,t)) \subset (1/2 - d(t), 1/2 + d(t)) \subset \Omega \textit{ for all } t \in [0, T^*],$$

and

$$d(t) \leqslant r_0 + C_0 t^{\frac{1}{n+2(s+1)}} \textit{ for all } t \in [0, T^*],$$

where $C_0 > 0$ ***depends on the initial data*** u_0 ***and on*** n***.***

Waiting time phenomenon

Theorem 2 .
Let us assume that $\Omega = (0,1)$, $s \in (0,1)$, $n \in (1,2)$, ***and***

$$\mathrm{supp}(h_0(\cdot)) \subseteq [1/2 - r_0, 1/2 + r_0],$$

and

$$\limsup_{\delta \to 0^+} \left(\delta^{-\gamma(2-n)} \fint_{\{|x-1/2| \leqslant r_0 + \delta\}} h_0^{2-n}\, dx \right) < \infty \text{ for } \gamma \geqslant \frac{2(s+1)}{n}.$$

Then there exists a time $T_0 = T_0(n, u_0)$ ***such that***

$$\mathrm{supp}(h(\cdot, t)) \subseteq [1/2 - r_0, 1/2 + r_0] \text{ for } t \in [0, T_0).$$

Moreover, the waiting time is estimated from below by

$$T_0 \geqslant C \left(\sup_{\delta > 0} \delta^{-\gamma(2-n)} \fint_{\{|x-1/2| \leqslant r_0 + \delta\}} h_0^{2-n}\, dx \right)^{-\frac{n}{2-n}}.$$

Example

If $u_0 \in C(\bar{\Omega})$ **then we may assume**

$$\max_{x\in(1/2,1/2+r_0+\delta]} h_0(x) \leqslant C|x-1/2-r_0|^{\gamma}$$

or

$$\max_{x\in[1/2-r_0-\delta,1/2)} h_0(x) \leqslant C|x-1/2+r_0|^{\gamma},$$

where $\gamma \geqslant \frac{2(s+1)}{n}$**,** $\delta \in (0, 1/2 - r_0)$**, and** $C > 0$**.**

Explicit waiting-time solution:

$$h(x,t) = \left(1 + c_{n,s}t\right)^{-1/n} x^{\frac{2(s+1)}{n}}$$

for $0 < n < 1 + s^{-1}$**.**

List of publications

- Nicola de Nitti, Roman M. Taranets, Interface propagation properties for a nonlocal thin-film equation, 21p., to appear in SIAM Journal on Mathematical Analysis
- Chugunova, M., Taranets, R., Vasylyeva, N. Initial-boundary value problems for conservative Kimura-type equations: solvability, asymptotic and conservation law. J. Evol. Equ. 23, 17 (2023).
- Roman M. Taranets, Hangjie Ji, Marina Chugunova, On travelling waves of a control-volume model for liquid films flowing down a fibre, arXiv:2301.02720v2, 47 p (2023) (to submit in DCDS-B)
- Belgacem Al-Azem, Marina Chugunova, Roman Taranets, Nataliya Vasylyeva, Qualitative analysis of solutions for a degenerate PDE model of epidemic spread dynamics, to submit in International Journal of Differential Equations.

Thank you!

THANK YOU FOR YOUR PATIENCE

THE END