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Qualitative theory of equations with nonstandard growth

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We study the qualitative properties of solutions to the equations with nonstandard growth:

$$div(|\nabla u|^{p(x)-2}\nabla u) = 0,\ p(x) > 1, \quad (1)$$

$$div(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = 0,\ \ a(x) \geq 0,\ 1 < p < q, \quad (2)$$

$$u_t - div(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = 0,\ \ a(x,t) \geq 0,\ 1 < p < q. \quad (3)$$

(1) - p(x)-Laplacian equation
(2) - double phase elliptic equation
(3) - double phase parabolic equation

Motivation

Double phase equations are used for describing the behavior of strongly anisotropic materials whose hardening properties—linked to the exponent ruling the growth of the gradient variable—drastically change with the point. The coefficient $a(\cdot)$ serves to regulate the mixture between two different materials, with p and q hardening, respectively.

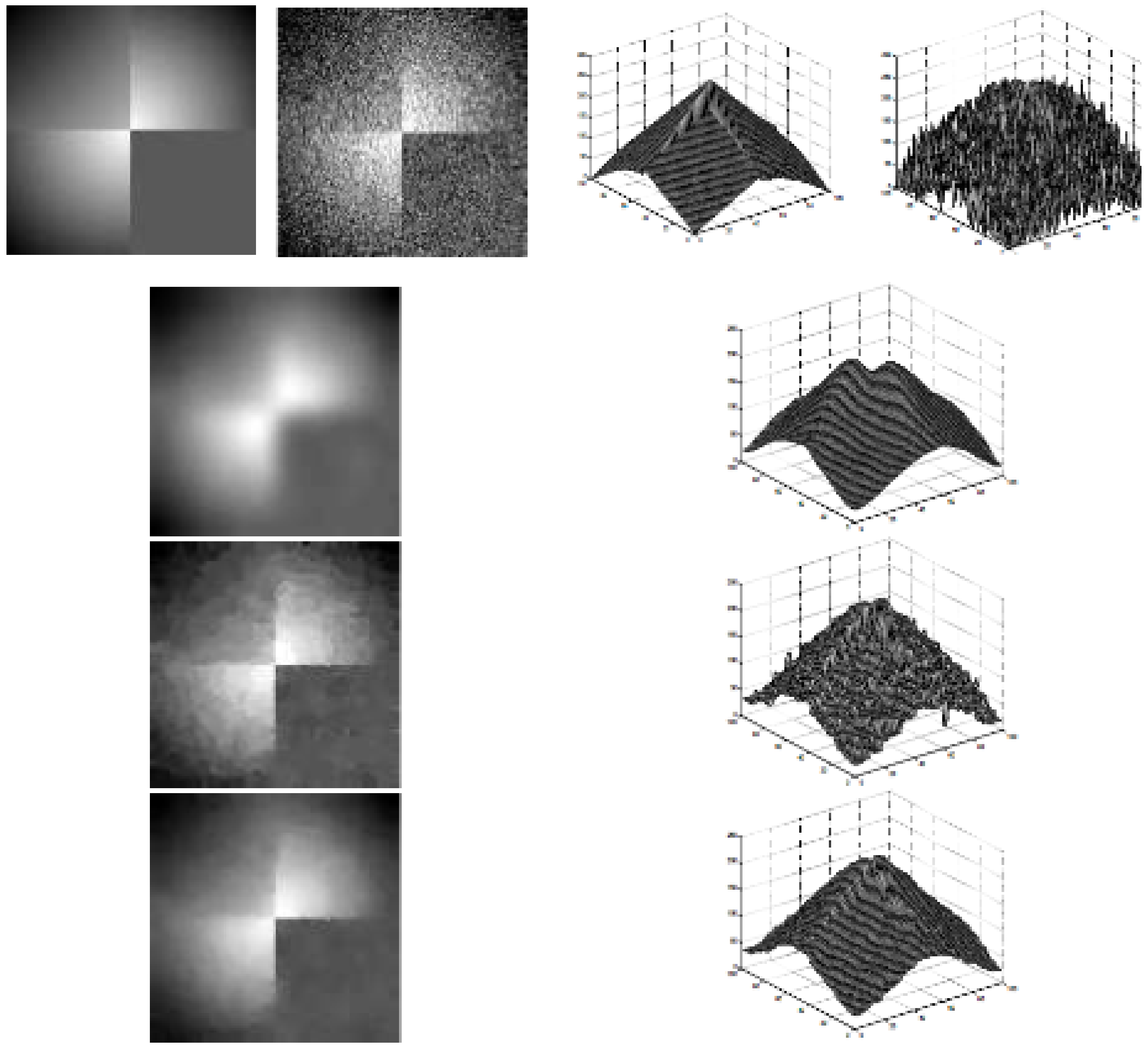
Motivation

u - original image
I - noisy image

$$\min \int_{\Omega} \left(|\nabla u|^2 + |u - I|^2\right) dx$$

$$\min \int_{\Omega} \left(|\nabla u| + |u - I|^2\right) dx$$

$$\min \int_{\Omega} \left(|\nabla u|^{p(x)} + |u - I|^2\right) dx$$



Results for Eq. (1)

Let u be a bounded weak solution of Eq. (1) and let condition

$$\underset{B_r(x_0)}{\operatorname{osc}} p(x) \leqslant \frac{L}{\log \frac{1}{r}}, \quad 0 < r < 1, \quad 0 < L < \infty, \tag{4}$$

be fulfilled, then u is Hölder continuous at point x_0.

Results for Eq. (1)

Let u be a bounded non-negative weak sub-solution to Eq. (1), let condition (4) be fulfilled. Then there exist positive numbers γ, $\bar{\gamma}_1$ depending only on n, p, K_1, K_2, M such that for any $\theta \in (0, p-1)$

$$\max_{B_{\frac{\rho}{2}}(x_0)} u \leqslant \gamma \left(|B_\rho(x_0)|^{-1} \int\limits_{B_\rho(x_0)} u^\theta \, dx \right)^{\frac{1}{\theta}} + \gamma \rho,$$

provided that $B_{16\rho}(x_0) \subset \Omega$ and

$$\frac{1}{\log \log \frac{1}{16\rho}} + \bar{\gamma}_1 L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leqslant 1.$$

Particularly, if u is a bounded non-negative weak solution to equation (1), then

$$\max_{B_{\frac{\rho}{2}}(x_0)} u \leqslant \gamma \left(\min_{B_{\frac{\rho}{2}}(x_0)} u + \rho \right),$$

provided that $B_{16\rho}(x_0) \subset \Omega$ and

$$\frac{1}{\log \log \frac{1}{16\rho}} + \max(\bar{\gamma}, \bar{\gamma}_1) L \frac{\log \log \frac{1}{16\rho}}{\log \frac{1}{16\rho}} \leqslant 1,$$

Result for Eq. (2)

Definition

Let $\Phi(x,v) := v^p + a(x)v^q$. We write $W^{1,\Phi(\cdot)}(\Omega)$ for the class of functions $u \in W^{1,1}(\Omega)$ with $\int\limits_{\Omega} \Phi(x,|\nabla u|)dx < \infty$ and we say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the elliptic class $DG^{\pm}_{\Phi}(\Omega)$ if $u \in W^{1,\Phi(\cdot)}(\Omega)$ and there exist numbers $c > 0$, $q > 1$ such that for any ball $B_{8r}(x_0) \subset \Omega$, any $k \in \mathbb{R}$ and any $\sigma \in (0,1)$ the following inequalities hold:

$$\int\limits_{A^{\pm}_{k,r(1-\sigma)}} \Phi(x,|\nabla u|)dx \leqslant \frac{c}{\sigma^q} \int\limits_{A^{\pm}_{k,r}} \Phi\left(x, \frac{(u-k)_{\pm}}{r}\right)dx,$$

here $(u-k)_{\pm} := \max\{\pm(u-k), 0\}$, $A^{\pm}_{k,r} := B_r(x_0) \cap \{(u-k)_{\pm} > 0\}$.

Result for Eq. (2)

We suppose that $\Phi(x,v):\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is a non-negative function such that for any $x\in\Omega$ the function $v\to\Phi(x,v)$ is increasing and $\lim\limits_{v\to 0}\Phi(x,v)=0$, $\lim\limits_{v\to+\infty}\Phi(x,v)=+\infty$. We also assume that

(Φ) There exist $1<p<q$ such that for $x\in\Omega$ and for $w\geqslant v>0$ there holds

$$\left(\frac{w}{v}\right)^p\leqslant\frac{\Phi(x,w)}{\Phi(x,v)}\leqslant\left(\frac{w}{v}\right)^q.$$

(Φ_λ) There exist $s>0$, $R>0$ and continuous, non-decreasing function $\lambda(r)\in(0,1)$ on the interval $(0,R)$, $\lim\limits_{r\to0}\lambda(r)=0$, $\lim\limits_{r\to0}\frac{r}{\lambda(r)}=0$, such that for any $B_r(x_0)\subset B_R(x_0)\subset\Omega$ and some $A>0$ there holds

$$\Phi^+_{B_r(x_0)}\left(\frac{\lambda(r)v}{r^{1+\frac{n}{s}}}\right)\leqslant A\,\Phi^-_{B_r(x_0)}\left(\frac{\lambda(r)v}{r^{1+\frac{n}{s}}}\right),\quad r^{1+\frac{n}{s}}\leqslant v\leqslant\lambda(r),$$

here

$$\Phi^+_{B_r(x_0)}(v):=\sup_{x\in B_r(x_0)}\Phi(x,v),\ \Phi^-_{B_r(x_0)}(v):=\inf_{x\in B_r(x_0)}\Phi(x,v),\quad v>0.$$

(λ) For any $0<r<\rho<R$ there holds

$$\lambda(r)\geqslant\lambda(\rho)\left(\frac{r}{\rho}\right)^b\ \text{ with some }\ b\geqslant0.$$

Conditions

Result for Eq. (2)

Let $u \in DG^{-}(\Omega)$, $u \geqslant 0$, let conditions (Φ), (Φ_λ), (λ) be fulfilled. Let $B_{8\rho}(x_0) \subset B_R(x_0) \subset \Omega$, let additionally $u \in L^s_{loc}(\Omega)$ with some $s > 0$ and $\left(\int_{B_{2\rho}(x_0)} u^s\right)^{\frac{1}{s}} \leqslant d$. Then there exists a positive constant C depending only on the known parameters and d, such that

$$\left(|B_\rho(x_0)|^{-1} \int\limits_{B_\rho(x_0)} (u+\rho)^\theta dx\right)^{\frac{1}{\theta}} \leqslant \frac{C}{\lambda(\rho)} \left(\inf_{B_{\frac{\rho}{2}}(x_0)} u + \rho\right),$$

where $\theta > 0$ is some fixed number depending only on the known data.

Result for Eq. (3)

Conditions

We assume that there exists positive continuous non-increasing function $\mu(r) \geqslant 1$ on the interval $(0,1)$, $\lim\limits_{r\to 0} \mu(r) r^{1-\bar{b}} = 0$ with some $\bar{b} \in (0,1)$ such that

$$| a(x,t) - a(y,\tau) | \leq A\mu(r) r^{q-p}, \ \ (x,t), (y,\tau) \in Q_{r,r}(x_0, t_0) \subset \Omega_T \qquad (5)$$

with some $A > 0$.

In addition, we suppose that the equation (3) is degenerate at the point (x_0, t_0) which means that there exists $K_3, R_0 > 0$ such that the function

$$\psi(x_0, t_0, v) := v^{p-2} + a(x_0, t_0) v^{q-2} \text{ is non-decreasing for } v \geqslant \frac{K_3}{R_0}. \quad (6)$$

Fix point $(x_0, t_0) \in \Omega_T$, let $u \in C(\Omega_T)$ be a positive bounded weak solution to Eq. (3) and let the conditions (5), (6) be fulfilled. Assume also that

$$a(x_0, t_0) > 0.$$

Then there exists $R > 0$, depending only on the data and $a(x_0, t_0)$, and there exist positive numbers c, C, depending only upon the data, such that for all $\rho \leqslant R^2$, either

$$u(x_0, t_0) \leqslant C\rho^{\frac{1}{2}},$$

or

$$u(x_0, t_0) \leqslant C \inf_{B_\rho(x_0)} u(\cdot, t)$$

with $t \in (t_0 + \frac{1}{2}\theta, t_0 + \theta)$, $\theta := \dfrac{\rho^2}{\psi\left(x_0, t_0, c\dfrac{u(x_0, t_0)}{\rho}\right)}$, provided that

$$Q_{\rho,\theta}(x_0, t_0) \subset Q_{\rho,\rho}(x_0, t_0) \subset Q_{R,R^2}(x_0, t_0) \subset Q_{8R,(8R)^2}(x_0, t_0) \subset \Omega_T.$$

Result for Eq. (3)

List of publications

PUBLISHED

Savchenko M., Skrypnik I., Yevgenieva Ye. Harnack's inequality for degenerate double phase parabolic equations under the non-logarithmic Zhikov's condition, Journal of Mathematical Sciences. - 2023. - V. 273, № 3. - P. 427 - 452.

ACCEPTED

Skrypnik I., Yevgenieva Ye. Harnack inequality for solutions of the p(x)-Laplace equation under the precise nonlogarithmic Zhikov's conditions, Calculus of Variations and Partial Differential Equations.

SUBMITTED

Savchenko M., Skrypnik I., Yevgenieva Ye. A note on the weak Harnack in equality for unbounded minimizers of elliptic functionals with generalized Orlicz growth, arXiv: 2304.04499 (Annali di Matematica Pura ed Applicata)