On a generalization of the Banach contraction principle

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The Banach fixed-point theorem

Let (X, d) be a metric space. Then a mapping $T\colon X \to X$ is called a *contraction mapping* on X if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{1}$$

for all $x, y \in X$.

Theorem (Banach, 1922)

Let (X, d) be a nonempty complete metric space with a contraction mapping $T\colon X \to X$. Then T admits a unique fixed point.

The Banach contraction principle has been generalized in many ways over the years. It is possible to distinguish at least three types of generalizations of such theorems:

- in the first case the contractive nature of the mapping is weakened;
- in the second case the topology is weakened;
- the third case is set-valued extensions.

Some well-known generalizations of the Banach fixed-point theorem

Theorem (Matkowski,1975)

Assume that (X, d) is a complete metric space and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that the sequence of iterates (φ^n) tends to zero pointwise on $\mathbb{R}_+ \setminus \{0\}$. If $T\colon X \to X$ is a mapping satisfying

$$d(Tx, Ty) \leqslant \varphi(d(x, y)) \tag{2}$$

for all $x, y \in X$, then it has a unique fixed point in X.

Theorem (Vulpe, I. M.; Ostrajkh, D.; Khojman, F., 1981)

Let (X, d) be a b-metric space with the constant $K \geqslant 1$ (i.e. $d(x, y) \leqslant K(d(x, z) + d(z, y))$) and let $T\colon X \to X$ be a mapping such that there exists a number $\alpha > 0$ for which

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{3}$$

for all $x, y \in X$. If the space (X, d) is complete and $\alpha K < 1$, then the mapping T has a unique fixed point.

Theorem (Nadler,1969)

Let (X, d) be a complete metric space and T be a multi-valued mapping on X such that $T(x)$ is a nonempty closed bounded subset for any $x \in X$. If there exists $\alpha \in (0, 1)$ such that

$$H(T(x), T(y)) \leqslant \alpha d(x, y), \tag{4}$$

for all $x, y \in X$, then there exists $z \in X$ such that $z \in T(z)$ (here H is the well-known Hausdorff metric).

Mappings contracting perimeters of triangles

Definition

Let (X, d) be a metric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{5}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Remark

Note that the requirement for $x, y, z \in X$ to be pairwise distinct is essential. Otherwise this definition is equivalent to the definition of contraction mapping.

Proposition

Mappings contracting perimeters of triangles are continuous.

Main theorem

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period* n if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x.

In particular, the point x is of prime period 2 if $T(T(x)) = x$ and $Tx \neq x$.

Theorem

Let (X, d), $|X| \geqslant 3$, be a complete metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles on X. Then T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most two.

Remark

Suppose that under the supposition of the theorem the mapping T has a fixed point x^* which is a limit of some iteration sequence $x_0, x_1 = Tx_0, x_2 = Tx_1, \dots$ such that $x_n \neq x^*$ for all $n = 1, 2, \dots$. Then x^* is a unique fixed point.

Corollary

Banach fixed-point theorem holds.

Elementary examples

1) The mapping T contracting perimeters of triangles which has exactly two fixed points.

Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T : X \to X$ be such that $Tx = x$, $Ty = y$ and $Tz = x$.

$P_1 = d(x, y) + d(y, z) + d(x, z) = 1 + 1 + 1 = 3$,

$P_2 = d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) = 1 + 1 + 0 = 2$ and

$$P_2 \leqslant \frac{2}{3} P_1.$$

Thus, (5) holds with $\alpha = \frac{2}{3} < 1$ and T does not have periodic points of prime period 2.

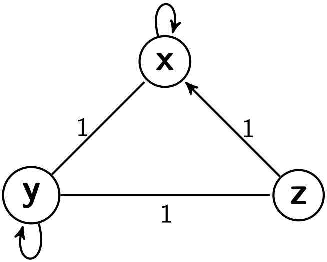

Figure: The space (X, d).

Elementary examples

2) The mapping T contracting perimeters of triangles which does not have any fixed point.

Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$, and let $T \colon X \to X$ be such that $Tx = y$, $Ty = x$ and $Tz = x$.

$P_1 = d(x, y) + d(y, z) + d(x, z) = 1 + 1 + 1 = 3$,

$P_2 = d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) = 1 + 0 + 1 = 2$ and

$$P_2 \leqslant \tfrac{2}{3} P_1.$$

Thus, (5) holds with $\alpha = \frac{2}{3} < 1$ and the points x and y are periodic points of prime period 2 (i.e., $T(Tx) = x$, $T(Ty) = y$).

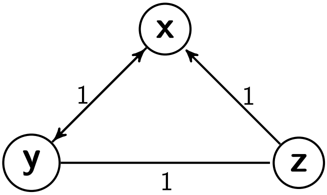

Figure: The space (X, d).

Spaces with accumulation points

Recall that for a given metric space X, a point $x \in X$ is said to be an *accumulation point* of X if every open ball centered at x contains infinitely many points of X.

Proposition

Let (X, d), $|X| \geqslant 3$, be a metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles with the coefficient α. If x is an accumulation point of X, then the inequality

$$d(Tx, Ty) \leqslant \alpha d(x, y)$$

holds for all points $y \in X$.

Corollary

Let (X, d), $|X| \geqslant 3$, be a metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles. If all points of X are accumulation points, then T is a contraction mapping.

Mapping contracting perimeters of triangles that is not a contraction mapping

Example

Let us construct an example of a mapping $T\colon X \to X$ contracting perimeters of triangles that is not a contraction mapping for a metric space X with $|X| = \aleph_0$. Let $X = \{x^*, x_0, x_1, \ldots\}$ and let a be positive real number.

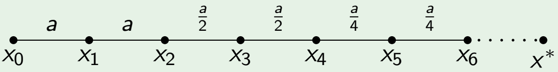

Figure: The points of the space (X, d) with consecutive distances between them.

The space is complete with the single accumulation point x^*.
Define a mapping $T\colon X \to X$ as $Tx_i = x_{i+1}$ for all $i = 0, 1, \ldots$ and $Tx^* = x^*$.
Since $d(x_{2n}, x_{2n+1}) = d(Tx_{2n}, Tx_{2n+1})$, $n = 0, 1, 2...$, we see that T is not a contraction mapping.

Strict definition of the metric d

Example

Define a metric d on X as follows:

$$d(x,y) = \begin{cases} a/2^{\lfloor i/2 \rfloor}, & \text{if } x = x_i, y = x_{i+1}, i = 0,1,2,\ldots, \\ d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j), & \text{if } x = x_i,\ y = x_j,\ i+1 < j, \\ 4a - d(x_0, x_i), & \text{if } x = x_i,\ y = x^*, \\ 0, & \text{if } x = y, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Publications

E. Petrov. Fixed point theorem for mappings contracting perimeters of triangles. *J. Fixed Point Theory Appl.*, 25, 2023. (paper no. 74). `https://arxiv.org/abs/2308.01003`

Thank you for your attention!