Institute of Applied Mathematics and Mechanics of NASU

Classical solutions to quasilinear time-fractional reaction-diffusion systems

Mykola Krasnoshchok, Sloviansk

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Statement of the problem

We deal with the Caputo fractional derivative of order $a \in (0,1)$ defined by

$$D_*^a w(x,t) = \frac{1}{\Gamma(1-a)} \frac{\partial}{\partial t} \int_0^t \frac{w(x,t) - w(x,0)}{(t-\theta)^a} d\theta,$$

Consider the following system

$$D_*^a u_i(x,t) - \mathcal{L}_i u_i(x,t) = f_i(u,x,t) \ \text{ in } Q_T, \tag{1}$$

$$u_i(x,0) = u_{0,i}(x) \text{ in } Q, i \in \{1,..,n\}, \tag{2}$$

$$\mathcal{M}_i u_i(x,t) = g_i(x,t) \text{ on } S_T, i \in \mathcal{I}_N, \tag{3}$$

$$u_i(x,t) = g_i(x,t) \text{ on } S_T, i \in \mathcal{I}_D, \tag{4}$$

where $\mathcal{L}_i$, $\mathcal{M}_i$ are strong elliptic and conormal boundary operators given by

$$\mathcal{L}_i u_i(x,t) = \sum_{k,l=1}^{d} a_{kl}^{(i)}(x,t) \frac{\partial^2 u_i(x,t)}{\partial x_k \partial x_l} + \sum_{k=1}^{d} a_k^{(i)}(x,t) \frac{\partial u_i(x,t)}{\partial x_k} + a_0^{(i)}(x,t) u_i(x,t),$$

$$\mathcal{M}_i u_i(x,t) = \sum_{k,l=1}^{d} a_{kl}^{(i)}(x,t) N_k(x) \frac{\partial u_i(x,t)}{\partial x_l} + c_0^{(i)}(x,t) u_i(x,t),$$

and $N(x) = (N_1(x), .., N_d(x))$ is the unit outward vector to Q.

Spaces

$w \in C^{2+\alpha}_{\beta,a}(\overline{Q}_T)$:

$$|w|^{(2+\alpha)}_{\beta,a,Q_T} = \sum_{|l|+2k=2} \sup_{t\in(0,T)} t^{\frac{2+\alpha-\beta}{2}a} \langle \left(D^a_{*,t}\right)^k D^l_x w \rangle^{(\alpha)}_{x,Q'_t}$$

$$+ \sum_{\alpha<|l|+2k<2+\alpha} \sup_{t\in(0,T)} t^{\frac{2+\alpha-\beta}{2}a} \langle \left(D^a_{*,t}\right)^k D^l_x w \rangle^{(\frac{2+\alpha-|l|-2k}{2}a)}_{x,Q'_t}$$

$$+ \sum_{\beta<|l|+2k<2+\alpha} \sup_{Q_T} t^{\frac{2k+|l|-\beta}{2}a} \left| \left(D^a_{*,t}\right)^k D^l_x w(x,t) \right| + |w|^{(\beta)}_{a,Q_T}.$$

here $Q'_t = Q \times (t/2, t)$.

Upper-lower solutions

Definition A pair of functions $\overline{u} = (\overline{u}_1, .., \overline{u}_n)$, $\underline{u} = (\underline{u}_1, .., \underline{u}_n)$ are called coupled upper-lower solutions of the system (1)-(2) if

(i) $\overline{u}_i, \underline{u}_i \in C^{2+\alpha}_{\beta,a}(\overline{Q}_T)$ for any $i \in \{1, .., n\}$,

(ii) $\overline{u}_i(x,t) \geq \underline{u}_i(x,t)$ for all $(x,t) \in Q_T, i \in \{1, .., n\}$,

iii) $D^a_* \overline{u}_i(x,t) - \mathcal{L}_i \overline{u}_i(x,t) \geq f_i(v_1, .., v_{i-1}, \overline{u}_i, v_{i+1}, .., v_n, x, t)$,
$\forall v_j (j \neq i) : \underline{u}_j(x,t) \leq v_j \leq \overline{u}_j(x,t)$ in $Q_T, i \in \{1, .., n\}$.

iv) $D^a_* \underline{u}_i(x,t) - \mathcal{L}_i \underline{u}_i(x,t) \leq f_i(v_1, .., v_{i-1}, \underline{u}_i, v_{i+1}, .., v_n, x, t)$,
$\forall v_j (j \neq i) : \underline{u}_j(x,t) \leq v_j \leq \overline{u}_j(x,t)$ in $Q_T, i \in \{1, .., n\}$.

v) $\mathcal{M}_i \overline{u} \geq g_i \geq \mathcal{M}_i \underline{u}_i, i \in \mathcal{I}_N;$ and $\overline{u} \geq g_i \geq \underline{u}_i,$ on $S_T, i \in \mathcal{I}_D$.

(vi) $\overline{u}_i(x,0) \geq u_{0,i}(x) \geq \underline{u}_i(x,0)$ in $Q, i \in \{1, .., n\}$.

Theorem. Let

i) $\overline{u}$, $\underline{u}$ be coupled upper-lower solutions of (1)-(2),

ii) $f_i(u,x,t) \in C^2(B_{n,\rho}; C^{\alpha}_{\beta-2}(\overline{Q}_T))$ for any $\rho > 0$ and there exists a constant $K_{\rho,i} > 0$ such that for any (x,t,u), $(x,t,v) \in B_{n,\rho}$,

$$|f_i(u,x,t) - f_i(v,x,t)| \leq K_{\rho,i}|u - v|, i \in \{1,..,n\}.$$

Then for arbitrary fixed $T > 0$ the problem (1)-(2) has a unique solution $u \in \bigoplus_{i=1}^{n} C^{2+\alpha}_{\beta,a}(\overline{Q}_T)$ and

$$\underline{u}_i(x,t) \leq u_i(x,t) \leq \overline{u}(x,t) \text{ in } Q_T,\ i \in \{1,..,n\}. \tag{5}$$

Applications

$$D_*^a u(x,t) - d_1 \Delta u(x,t) = -r_1 u^m(x,t) v^n(x,t), \text{ in } Q_T,$$
$$D_*^a v(x,t) - d_1 \Delta v(x,t) = -r_2 u^m(x,t) v^n(x,t), \text{ in } Q_T, \tag{6}$$

$$\frac{\partial u}{\partial N}(x,t) = \frac{\partial v}{\partial N}(x,t) = 0, \text{ on } S_T, \tag{7}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \text{ in } Q, \tag{8}$$

where d_1, d_2, r_1, r_2 are fixed positive parameters, m, $n \geq 2$.

Assumptions

Assume that

$$u_0, v_0 \in C^\beta(\overline{Q}),$$

$$0 \le m_1 \le u_0(x) \le M_1, \quad 0 \le m_2 \le v_0(x) \le M_2 \quad \text{in } Q. \tag{9}$$

$$m_1 > \frac{r_1}{r_2} M_2 > 0. \tag{10}$$

Large time behaviour

Proposition Let the assumption (9), (10) hold. Then there exist the unique solution (u, v) to the system (6)-(8) such that

$$u, v \in C^{2+\alpha}_{\beta,a}(\overline{Q}_T),$$

$$m_1 - \frac{r_1}{r_2} M_2 + \frac{\mathrm{C}_1}{1 + t^{\frac{a}{n}}} \leq u(x,t) \leq M_1 - \frac{r_1}{r_2} m_2 + \frac{C_2}{1 + t^{\frac{a}{n}}}$$

$$\text{for any } (x,t) \in Q_T,$$

$$\frac{\mathrm{C}_1}{1 + t^{\frac{a}{n}}} \leq v(x,t) \leq \frac{C_2}{1 + t^{\frac{a}{n}}} \text{ for any } (x,t) \in Q_T,$$

where the constants C_1, C_2 depend on m, n, and r_i, m_i, M_i, $i = 1, 2$.

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THANK YOU

FOR YOUR PATIENCE!