Homeomorphisms between Finsler manifolds satisfying the Rohde condition

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We study the main features of homeomorphisms between two Finsler manifolds satisfying the Rohde condition, i.e. the difference between the moduli of geodesic rings in the image and its preimage is uniformly bounded from below and above. Some equivalent conditions for such a characterization are established. Also we investigate the absolute continuity and the relations to the class of mappings with finite are distortion. As applications, the boundary correspondence results for homeomorphisms satisfying the Rohde condition between regular domains are also presented.

In [Rohde], it was established that for a proper subclass of quasiconformal mappings, namely L-bi-Lipschitz mappings of the whole $\mathbb{R}^n$, the difference between the moduli of rings in the image and preimage is restricted by a constant from below and above. The following inequality

$$|M(f(A)) - M(A)| \leq M, \tag{1}$$

will be called the *Rohde condition*.

[Rohde] S. Rohde, *Bilipschitz maps and the modulus of rings*. Ann. Acad. Sci. Fenn. Math. **22** (1997), no. 2, 465–474.

The following theorem provides the equivalence of two conditions for bi-Lipschitz mappings between Finsler manifolds.

Theorem. *Let $f : \mathbb{M} \to \mathbb{M}'$ is a homeomorphism with two fixed points. Suppose that there is a constant M such that the Rohde condition (1) holds for all geodesic rings A in $\mathbb{M}$. Then it is equivalent that there exists a constant $k \geq 1$ such that*

$$\frac{1}{k} \leq l(x, f) \leq L(x, f) \leq k \tag{2}$$

holds for all $x \in \mathbb{M}$. Moreover, f is an L-bi-Lipschitz homeomorphism in $\mathbb{M}$ with L depending on M and n only.

Recall that a metric space $\mathbb{M}$ is called *hyperconvex* if $\cap_{\alpha\in\Lambda}\overline{B}(x_\alpha, r_\alpha) \neq \varnothing$ for any collection of points $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\mathbb{M}$ and of positive numbers $\{r_\alpha\}_{\alpha\in\Lambda}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for any α and β in Λ; see [EN].

Our next result relates to the absolute continuity in measure.

Theorem. *Let $\mathbb{M}$ and $\mathbb{M}'$ be hyperconvex spaces. Suppose that $f : \mathbb{M} \to \mathbb{M}'$ is a homeomorphism with $L(x, f) \leq M$ and $l(x, f) \geq 1/M$ for all $x \in \mathbb{M}$, where M is a constant. Then for any set E in $D \subset \mathbb{M}$, f possesses the Lusin (N)- and (N^{-1})-properties with respect the k-dimensional Hausdorff measure for all $k = 1, ..., n$, i.e. $H^k(E) = 0$ if and only if $H^k(f(E)) = 0$.*

[EN] R. Espinola and A. Nicolae, *Continuous selections of Lipschitz extensions in metric spaces*. Rev. Mat. Complut. **28** (2015), no. 3, 741–759.

Theorem. *Let* $f : \mathbb{M} \to \mathbb{M}'$ *be a homeomorphism. Suppose that there exists a constant* $k \geq 1$ *such that (2) holds for all* $x \in \mathbb{M}$, *then* f *is a* K*-quasiconformal mapping, and, moreover, the inequality*

$$k^{-2}\mu'_f(x) \leq [L(x,f)]^n \leq k^2 [H_f(x)]^n \mu'_f(x) \tag{3}$$

holds for almost every x *in* $\mathbb{M}$.

Theorem. *Suppose that* $f : \mathbb{M} \to \mathbb{M}'$ *is a homeomorphism such that there exists a constant* $k \geq 1$ *for which the inequality (2) holds for all* $x \in \mathbb{M}$. *Then there exists a constant* C_0 *depending only on* L, n *and on the regularity constants of* $\mathbb{M}$ *and* $\mathbb{M}'$ *so that*

$$\frac{1}{C_0} M(\Gamma) \leq M(f(\Gamma)) \leq C_0 M(\Gamma) \tag{4}$$

holds for all curve families Γ *in* $\mathbb{M}$, *where* L *from definition of* L*-Lipschitz.*

A metric space is called C-*quasiconvex* if there is a constant $C \geq 1$ so that every pair of points x and y in the space can be joined by a curve whose length does not exceed $C\,d(x,y)$. A space is *locally quasiconvex* if every point in it has a quasiconvex neighborhood. We also say that a set $G \subset \mathbb{M}$ is C-*quasiconvex*, $1 \leq C < \infty$, if each pair of points $x, y \in G \setminus \{\infty\}$ can be joined in G by a rectifiable curve γ whose length does not exceed $C\,d(x,y)$.

The following theorem provides sufficient conditions which guarantee for a K-quasiconformal mapping to be an FMD-homeomorphism.

Theorem. *Let D and D' be two C-quasiconvex domains in $\mathbb{M}$ and $\mathbb{M}'$, respectively, $f : D \to D'$ be a K-quasiconformal homeomorphism. Then f is a locally FMD-homeomorphism in D.*

The following two theorems provide sufficient conditions which guarantee for a K-quasiconformal mapping (or a homeomorphism satisfying the Rohde condition) to be of finite area distortion.

Theorem. *Let $\mathbb{M}'$ be a hyperconvex space and $f : \mathbb{M} \to \mathbb{M}'$ be a K-quasiconformal mapping. Assume that there is a constant M such that (1) holds for all geodesic rings A centered at points of $D \subset \mathbb{M}$ with the property that both boundary spheres meet D. Then the restriction of f to D is an FAD–homeomorphism.*

Theorem. *Suppose that $\mathbb{M}'$ is a hyperconvex space and $f : \mathbb{M} \to \mathbb{M}'$ is a homeomorphism. Assume that there is a constant M such that (1) holds for all geodesic rings A in $\mathbb{M}$. Then f is an FAD-homeomorphism in D.*

We say that a mapping $f:\Omega\to\mathbb{M}^m$ is n,λ-*absolutely continuous* (shortly AC^n_λ) if for each $\varepsilon>0$ there exists $\delta>0$ such that for every pairwise disjoint finite collection $\{B(x_j,r_j)\}$ of balls in $\Omega\subset D$, we have

$$\sum_j \sigma_\Phi(B(x_j,r_j)) < \delta \qquad \Longrightarrow \qquad \sum_j \left({}_{B(x_j,\lambda r_j)}f\right)^n < \varepsilon\,, \tag{5}$$

where $0<\lambda<1$. This definition can be found in [He].

[He] S. Hencl, *On the notions of absolute continuity for functions of several variables*. Fund. Math. **173** (2002), no. 2, 175–189.

Theorem. *Let* $f : \mathbb{M} \to \mathbb{M}'$ *be a homeomorphism with two fixed points. If there is a constant* M *such that (1) holds for all geodesic rings* A *in* $\mathbb{M}$, *then* $f \in \mathrm{AC}^n_\lambda$.

Theorem. *Let* $f : \mathbb{M} \to \mathbb{M}'$ *be a* K*-quasiconformal homeomorphism with two fixed points. Assume that there is a constant* M *such that (1) holds for all geodesic rings* A *centered at points of* $D \subset \mathbb{M}$ *with the property that both boundary spheres meet* D. *Then* $f \in \mathrm{AC}^{n,p}_\lambda(D)$, *for some* $p > n$.

List of publications:

1. Elena Afanas'eva, Homeomorphisms between Finsler manifolds satisfying the Rohde condition. Ukrainian Mathematical Bulletin, Vol. 20, No. 3, 2023.